

TEST MAP CHARACTERIZATIONS OF LOCAL PROPERTIES OF FUNDAMENTAL GROUPS

JEREMY BRAZAS AND HANSPETER FISCHER

ABSTRACT. Local properties of the fundamental group of a path-connected topological space can pose obstructions to the applicability of covering space theory. A generalized covering map is a generalization of the classical notion of covering map defined in terms of unique lifting properties. The existence of generalized covering maps depends entirely on the verification of the unique path lifting property for a standard covering construction. Given any path-connected metric space X , and a subgroup $H \leq \pi_1(X, x_0)$, we characterize the unique path lifting property relative to H in terms of a new closure operator on the π_1 -subgroup lattice that is induced by maps from a fixed “test” domain into X . Using this test map framework, we develop a unified approach to comparing the existence of generalized coverings with a number of related properties.

1. INTRODUCTION AND PRELIMINARIES

1.1. Introduction. According to classical covering space theory [26], if a path-connected space X is locally path connected and semilocally simply connected, then for every subgroup $H \leq \pi_1(X, x_0)$ of the fundamental group at basepoint $x_0 \in X$, there is a covering map $p : (Y, y_0) \rightarrow (X, x_0)$ such that H is the image of the induced homomorphism $p_\# : \pi_1(Y, y_0) \rightarrow \pi_1(X, x_0)$, and conjugates of H correspond to equivalent covering maps. In particular, a universal covering over X exists. Since all “small” loops are null-homotopic in X , the natural topologies typically considered on $\pi_1(X, x_0)$ [4, 6, 7, 28] are equivalent to the discrete topology, i.e. all subgroups of $\pi_1(X, x_0)$ are both open and closed. In this sense, the subgroup lattice of $\pi_1(X, x_0)$ is independent of the local structure of X .

When a space X is not semilocally simply connected, the situation is more delicate, since the algebraic structure of $\pi_1(X, x_0)$ may depend heavily on the local topology of X . The variety of possible complications has given rise to the introduction of a number of important properties that a space X might or might not satisfy, including:

- (1) Homotopically Hausdorff [2, 9] and its relative version [24],
- (2) Strongly homotopically Hausdorff [10],
- (3) (*transfinite products*) Every homomorphism $f_\# : \pi_1(\mathbb{H}, b_0) \rightarrow \pi_1(X, x_0)$ induced by a map $f : \mathbb{H} \rightarrow X$ on the Hawaiian earring is uniquely determined by its values $f_\#([\ell_n])$ on the individual loops ℓ_n of \mathbb{H} ,
- (4) Existence of generalized universal and intermediate coverings [24],
- (5) Homotopically path Hausdorff [22] and its relative version [6],
- (6) ($1-UV_0$) For every $x \in X$ and every neighborhood U of x there is an open set V in X with $x \in V \subseteq U$ and such that for every map $f : D^2 \rightarrow X$

from the unit disk with $f(\partial D^2) \subseteq V$, there is a map $g : D^2 \rightarrow U$ with $f|_{\partial D^2} = g|_{\partial D^2}$ [10],

- (7) (π_1 -*shape injectivity*) The canonical homomorphism $\pi_1(X, x_0) \rightarrow \check{\pi}_1(X, x_0)$ to the first shape homotopy group is injective [24].

The above properties are not listed in any particular order. Some of them originated from the unpublished notes [29]. See [22] for a diagram comparing properties (1),(2),(4),(5), and (7). Property (3) plays a key role in Eda's remarkable classification of homotopy types of one-dimensional Peano continua according to the isomorphism types of their fundamental groups [17]. Property (6) first appeared in [10].

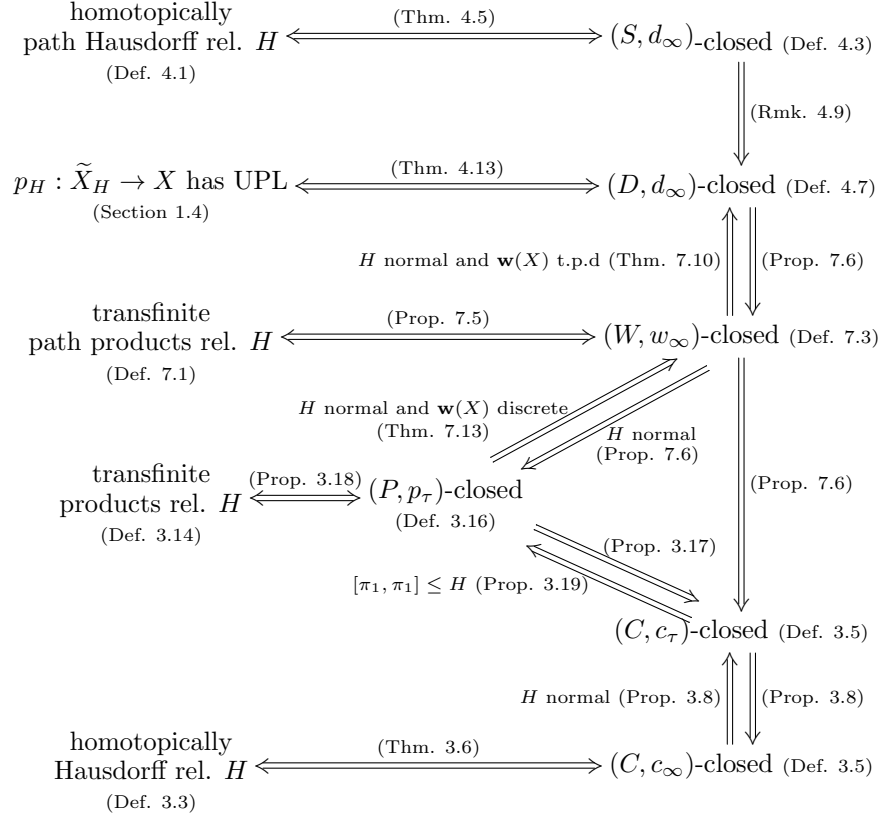
The primary purpose of this paper is to provide a unified approach to comparing local properties of fundamental groups such as those above. We are particularly motivated by the fact that even when X fails to admit a traditional universal covering, it is often the case that X admits a generalized universal covering in the sense of [24], which acts in many ways as a suitable replacement. A generalized covering map is characterized only by its lifting properties and need not be a local homeomorphism. For instance, in the case that X is a one-dimensional Peano continuum (e.g. the Hawaiian earring, Sierpinski Carpet, or Menger curve), a generalized universal covering exists, inherits the structure of an \mathbb{R} -tree, and functions as a generalized Caley graph for the fundamental group $\pi_1(X, x_0)$ [25]. Other spaces which admit generalized universal coverings include subsets of closed surfaces (including all planar sets) [23] and certain trees of manifolds [21] such as the Pontryagin surface Π_2 .

For a given space X , there may be many intermediate subgroups $H \leq \pi_1(X, x_0)$ which do not correspond to a covering map but which correspond to a generalized covering as defined under the name **lpc**₀-covering in [3]. Some examples of generalized *regular* coverings corresponding to normal subgroups $N \trianglelefteq \pi_1(X, x_0)$, namely intersections of Spanier groups, appear in [24]. The same normal subgroups also appear in [7] with an equivalent construction. In the current paper, we consider the existence of generalized coverings relative to an arbitrary subgroup H of $\pi_1(X, x_0)$.

Roughly speaking, in order to verify any one of the properties (1)-(7), it is necessary to detect the existence of a specific homotopy given a certain, possibly infinite, arrangement of paths. In this paper, we formalize this viewpoint by characterizing the subgroup-relative versions of many of these properties. Our characterizations are stated in terms of set-theoretic closure operations on the subgroup lattice of $\pi_1(X, x_0)$. For a given property, we identify a based *test space* (\mathbb{T}, t_0) , a subgroup $T \leq \pi_1(\mathbb{T}, t_0)$, and an element $g \in \pi_1(\mathbb{T}, t_0)$. We call (T, g) a *closure pair* for (\mathbb{T}, t_0) and declare a subgroup $H \leq \pi_1(X, x_0)$ to be (T, g) -*closed* if it satisfies the following criterion: for every map $f : (\mathbb{T}, t_0) \rightarrow (X, x_0)$ such that $f_{\#}(T) \leq H$, we also have $f_{\#}(g) \in H$. Using this test map criterion, we characterize properties (1),(3),(4),(5) and compare them to a variety of other properties, including some new intermediate properties.

1.2. Results. The following diagram may serve as a reference for many of the results and definitions in this paper. It connects the relevant properties of a path-connected metric space X and closure properties of a subgroup $H \leq \pi_1(X, x_0)$. If an extra assumption is required, it appears next to the corresponding arrow. For example, “**w**(X) t.p.d.” denotes the property that the subset of points at which X fails to be semilocally simply connected is a totally path-disconnected subspace of X

and $[\pi_1, \pi_1] \leq H$ indicates that H contains the commutator subgroup of $\pi_1(X, x_0)$. For the non-reversibility of some of the implications see Corollary 3.13: (C, c_∞) -closed $\nRightarrow (C, c_\tau)$ -closed, Theorem 3.22: (C, c_τ) -closed $\nRightarrow (P, p_\tau)$ -closed, Example 5.2: (C, c_τ) -closed $\nRightarrow (D, d_\infty)$ -closed, and Example 5.6: (D, d_∞) -closed $\nRightarrow (S, d_\infty)$ -closed.



Equipped with this chart, we identify new types of subgroups that correspond to intermediate generalized coverings (Theorem 5.4 and Corollaries 7.14, 7.15) and shed more light on the relative position of the commutator subgroup of $\pi_1(\mathbb{H}, b_0)$ (Example 3.10). We also extend the existence of generalized universal coverings for Peano continua with residually n -slender fundamental group to all metric spaces (Corollary 6.5).

Property (6) is not an invariant of homotopy type but is an important property held by one-dimensional [12] and planar spaces [23] and is known to imply the homotopically Hausdorff property for metric spaces [10]. We improve this result by showing that every metric space with the $1\text{-}UV_0$ property admits a generalized universal covering space (Theorem 6.9).

1.3. Notational considerations. Throughout this paper, X will denote a path-connected topological space with basepoint $x_0 \in X$ and H will denote a subgroup of the fundamental group $\pi_1(X, x_0)$. A *map* $f : X \rightarrow Y$ means a continuous function and $f_\# : \pi_1(X, x) \rightarrow \pi_1(Y, y)$ will denote the homomorphism induced by f on fundamental groups when $f(x) = y$.

If $\alpha : [0, 1] \rightarrow X$ is a path, then $\alpha^-(t) = \alpha(1 - t)$ is the reverse path. If $\alpha, \beta : [0, 1] \rightarrow X$ are paths such that $\alpha(1) = \beta(0)$, then $\alpha \cdot \beta$ denotes the usual concatenation of paths. More generally, if $\alpha_1, \alpha_2, \dots, \alpha_n$ is a sequence of paths such that $\alpha_j(1) = \alpha_{j+1}(0)$ for each j , then $\prod_{j=1}^n \alpha_j = \alpha_1 \cdot \alpha_2 \cdots \alpha_n$ is the path defined as α_j on $[\frac{j-1}{n}, \frac{j}{n}]$. The constant path at $x \in X$ is denoted by c_x . If $[a, b], [c, d] \subseteq [0, 1]$ and $\gamma : [a, b] \rightarrow X$, $\delta : [c, d] \rightarrow X$ are maps, we write $\gamma \equiv \delta$ if $\gamma = \delta \circ \phi$ for some increasing homeomorphism $\phi : [a, b] \rightarrow [c, d]$; if ϕ is linear and if it does not create confusion, we will identify γ and δ .

A path $\alpha : [a, b] \rightarrow X$ is *reduced* if whenever $a \leq s < t \leq b$ with $\alpha(s) = \alpha(t)$, the loop $\alpha|_{[s, t]}$ is not null-homotopic. Note that a constant path $\alpha : [a, b] \rightarrow X$ is reduced if and only if α is degenerate, that is, if $a = b$. If X is a one-dimensional metric space, then every path $\alpha : [0, 1] \rightarrow X$ is homotopic (rel. endpoints) within $\alpha([0, 1])$ to either a constant path or a reduced path, which is unique up to reparameterization [15].

For a given space X , let $P(X)$ denote space of paths in X with the compact-open topology generated by the subbasic sets $\langle K, U \rangle = \{\alpha | \alpha(K) \subseteq U\}$ where $K \subseteq [0, 1]$ is compact and $U \subseteq X$ is open. A convenient basis for the compact-open topology is given by neighborhoods of the form $\bigcap_{j=1}^{2^n} \langle [\frac{j-1}{2^n}, \frac{j}{2^n}], U_j \rangle$ where $n \in \mathbb{N}$. It is well-known that if (X, d) is a metric space, then the compact-open topology on $P(X)$ agrees with the topology of uniform convergence. For given $x \in X$, let $P(X, x) \subseteq P(X)$ denote the subspace of paths which start at x and $\Omega(X, x)$ denote the subspace of loops based at x .

If $H \leq \pi_1(X, x_0)$ is a subgroup and $\alpha : [0, 1] \rightarrow X$ is a path from $\alpha(0) = x_0$ to $\alpha(1) = x$, let $H^\alpha = [\alpha^-]H[\alpha] \leq \pi_1(X, x)$ denote the conjugate subgroup under basepoint change.

1.4. Generalized covering maps and the unique path lifting property. In [24], Fischer and Zastrow initially define the notion of *generalized universal covering* and *generalized regular covering* relative to a normal subgroup $N \trianglelefteq \pi_1(X, x_0)$. The following definition extends these definitions to general subgroups of $\pi_1(X, x_0)$; it appears in [3] under the name “**lpc**-covering.”

Definition 1.1. A map $p : \widehat{X} \rightarrow X$ is a *generalized covering map* if

- (1) \widehat{X} is nonempty, path connected, and locally path connected,
- (2) for every path-connected, locally path-connected space Y , point $\widehat{x} \in \widehat{X}$, and based map $f : (Y, y) \rightarrow (X, p(\widehat{x}))$ such that $f_\#(\pi_1(Y, y)) \leq p_\#(\pi_1(\widehat{X}, \widehat{x}))$, there is a unique map $\widehat{f} : (Y, y) \rightarrow (\widehat{X}, \widehat{x})$ such that $p \circ \widehat{f} = f$.

If $p_\#(\pi_1(\widehat{X}, \widehat{x}))$ is a normal subgroup of $\pi_1(X, x_0)$, we call p a *generalized regular covering map*. If \widehat{X} is simply-connected, we call p a *generalized universal covering map*.

Whenever a generalized covering $p : \widehat{X} \rightarrow X$ such that $p(\widehat{x}) = x_0$ exists, it is characterized up to equivalence by the conjugacy class of the subgroup $H = p_\#(\pi_1(\widehat{X}, \widehat{x})) \leq \pi_1(X, x_0)$. Since $[0, 1]$ is simply-connected, it is clear that for each point $\widehat{x} \in \widehat{X}$, every path $\alpha \in P(X, p(\widehat{x}))$ has a unique lift $\widehat{\alpha} \in P(\widehat{X}, \widehat{x})$ such that $p \circ \widehat{\alpha} = \alpha$. In particular p has the unique path lifting property:

Definition 1.2. A map $f : X \rightarrow Y$ has the *unique path lifting property* if whenever $\alpha, \beta : [0, 1] \rightarrow X$ are paths with $\alpha(0) = \beta(0)$ and $f \circ \alpha = f \circ \beta$, then $\alpha = \beta$.

In the attempt to construct generalized covering spaces, one is led to the following standard construction [26]. We refer to [24] for proofs of the basic properties. Given a subgroup $H \leq \pi_1(X, x_0)$, let $\tilde{X}_H = P(X, x_0)/\sim$ where $\alpha \sim \beta$ iff $\alpha(1) = \beta(1)$ and $[\alpha \cdot \beta^-] \in H$. The equivalence class of α is denoted $H[\alpha]$. We give \tilde{X}_H the so-called *standard topology* generated by the sets

$$B(H[\alpha], U) = \{H[\alpha \cdot \epsilon] \mid \epsilon([0, 1]) \subseteq U\}$$

where U is an open neighborhood of $\alpha(1)$ in X . This topology is called the *whisker topology* by some authors [7, 28]. The space \tilde{X}_H is path connected and locally path connected by construction and if $H[\beta] \in B(H[\alpha], U)$, then $B(H[\alpha], U) = B(H[\beta], U)$. We take the class $\tilde{x}_H = H[c_{x_0}]$ of the constant path to be the basepoint of \tilde{X}_H . In the case that $H = 1$ is the trivial subgroup, we simply write \tilde{X} and \tilde{x} for this space and its basepoint. Let $p_H : \tilde{X}_H \rightarrow X$ denote the endpoint projection map defined as $p_H(H[\alpha]) = \alpha(1)$. Since p_H maps $B(H[\alpha], U)$ onto the path-component of U which contains $\alpha(1)$, p_H is an open map if and only if X is locally path connected.

For a point $x \in X$, let \mathcal{T}_x be the set of all open neighborhoods of x in X . For $U \in \mathcal{T}_x$ and a path $\alpha : [0, 1] \rightarrow X$ from x_0 to x , consider the subgroup $\pi(\alpha, U) = \{[\alpha \cdot \delta \cdot \alpha^-] \mid \delta \in \Omega(U, \alpha(1))\} \leq \pi_1(X, x_0)$. Let $\pi(x, U) = \langle \pi(\alpha, U) \mid \alpha(1) = x \rangle$ be the subgroup generated by all subgroups $\pi(\alpha, U)$ in $\pi_1(X, x_0)$ with $\alpha(1) = x$ and note that $\pi(x, U)$ is a normal subgroup of $\pi_1(X, x_0)$.

Theorem 1.3. [26] *Suppose X is locally path connected. Then $p_H : \tilde{X}_H \rightarrow X$ is a covering map in the classical sense if and only if for every $x \in X$, there is a $U \in \mathcal{T}_x$ such that $\pi(x, U) \leq H$,*

If X is semilocally simply connected, then for every x , there is a $U \in \mathcal{T}_x$ such that $\pi(x, U) = 1$, so p_H is a covering map for every subgroup $H \leq \pi_1(X, x_0)$. The standard lifting properties of covering maps illustrate that every covering map is a generalized covering map.

Even when p_H is not a covering map, it may still be a generalized covering. The authors of [24] show that p_H is a generalized covering map whenever it has the unique path lifting property. Indeed, every path $\alpha : ([0, 1], 0) \rightarrow (X, x_0)$ has a continuous *standard lift* $\tilde{\alpha}_{\mathcal{S}} : [0, 1] \rightarrow \tilde{X}_H$ starting at \tilde{x}_H defined as $\tilde{\alpha}_{\mathcal{S}}(t) = H[\alpha_t]$ where $\alpha_t(s) = \alpha(st)$. Thus, to verify whether or not p_H is a generalized covering map, it is necessary and sufficient to verify that for each path $\alpha \in P(X, x_0)$, the standard lift $\tilde{\alpha}_{\mathcal{S}}$ is the *only* lift of α starting at \tilde{x}_H .

On the other hand, if $p : (\hat{X}, \hat{x}) \rightarrow (X, x_0)$ is a generalized covering map such that $p_{\#}(\pi_1(\hat{X}, \hat{x})) = H$, then there is a homeomorphism $q : (\hat{X}, \hat{x}) \rightarrow (\tilde{X}_H, \tilde{x}_H)$ such that $p_H \circ q = p$ [3]. This means that the topology of any generalized covering space must be equivalent to the standard topology. These observations are summarized in the following lemma.

Lemma 1.4. [3, Theorem 5.11] *For any subgroup $H \leq \pi_1(X, x_0)$, the following are equivalent:*

- (1) p_H has the unique path lifting property,
- (2) p_H is a generalized covering map,
- (3) $(p_H)_{\#}(\pi_1(\tilde{X}_H, \tilde{x}_H)) = H$,

- (4) X admits a generalized covering $p : (\widehat{X}, \widehat{x}) \rightarrow (X, x_0)$ such that $p_{\#}(\pi_1(\widehat{X}, \widehat{x})) = H$.

2. TEST MAPS, CLOSURE PAIRS, AND CLOSURE OPERATORS

Definition 2.1. Suppose (\mathbb{T}, t_0) is a based space, $T \leq \pi_1(\mathbb{T}, t_0)$ a subgroup, and $g \in \pi_1(\mathbb{T}, t_0)$. A subgroup $H \leq \pi_1(X, x_0)$ is (T, g) -closed if for every map $f : (\mathbb{T}, t_0) \rightarrow (X, x_0)$ such that $f_{\#}(T) \leq H$, we also have $f_{\#}(g) \in H$. We often refer to \mathbb{T} as a *test space* and (T, g) as a *closure pair* for (\mathbb{T}, t_0) .

Observe that the set of (T, g) -closed subgroups of $\pi_1(X, x_0)$ is closed under intersection and therefore forms a complete lattice. For any $H \leq \pi_1(X, x_0)$, we may define the (T, g) -closure of H as

$$cl_{T,g}(H) = \bigcap \{K \leq \pi_1(X, x_0) \mid K \text{ is } (T, g)\text{-closed and } H \leq K\}.$$

Note that $cl_{T,g}(H) = H$ if and only if H is (T, g) -closed. Moreover, $cl_{T,g}$ is a set-theoretic closure operator on the lattice of subgroups of $\pi_1(X, x_0)$ in the sense that $H \leq cl_{T,g}(H)$, $H \leq K$ implies $cl_{T,g}(H) \leq cl_{T,g}(K)$, and $cl_{T,g}(cl_{T,g}(H)) = cl_{T,g}(H)$.

Proposition 2.2. *If $f : (X, x_0) \rightarrow (Y, y_0)$ is a map and $H \leq \pi_1(X, x_0)$, then $f_{\#}(cl_{T,g}(H)) \leq cl_{T,g}(f_{\#}(H))$.*

Proof. First, observe that if $K \leq \pi_1(Y, y_0)$ is (T, g) -closed, then so is $f_{\#}^{-1}(K) \leq \pi_1(X, x_0)$. Let $k \in cl_{T,g}(H)$ and $K \leq \pi_1(Y, y_0)$ be any (T, g) -closed subgroup such that $f_{\#}(H) \leq K$. It suffices to show $f_{\#}(k) \in K$. Since $f_{\#}^{-1}(K)$ is (T, g) -closed and $H \leq f_{\#}^{-1}(K)$, we have $k \in cl_{T,g}(H) \leq f_{\#}^{-1}(K)$. Therefore, $f_{\#}(k) \in K$. \square

Proposition 2.3. *Suppose (T, g) and (T', g') are closure pairs for (\mathbb{T}, t_0) and (\mathbb{T}', t'_0) respectively. Then the following are equivalent:*

- (1) $g' \in cl_{T,g}(T')$,
- (2) for any space (X, x_0) and subgroup $H \leq \pi_1(X, x_0)$, H is (T', g') -closed whenever H is (T, g) -closed,
- (3) for any space (X, x_0) and subgroup $H \leq \pi_1(X, x_0)$, $cl_{T',g'}(H) \leq cl_{T,g}(H)$.

Proof. (1) \Rightarrow (2) Suppose $g' \in cl_{T,g}(T')$ and that $H \leq \pi_1(X, x_0)$ is (T, g) -closed. Let $k : (\mathbb{T}', t'_0) \rightarrow (X, x_0)$ be a map such that $k_{\#}(T') \leq H$. By Proposition 2.2 and monotonicity, we have $k_{\#}(cl_{T,g}(T')) \leq cl_{T,g}(k_{\#}(T')) \leq cl_{T,g}(H)$. Since $g' \in cl_{T,g}(T')$, it follows that $k_{\#}(g') \in cl_{T,g}(H) = H$. This proves H is (\mathbb{T}', g') -closed. (2) \Rightarrow (3) This follows directly from the definition of the closure operator. (3) \Rightarrow (1) First, note that $g' \in cl_{T',g'}(T')$. Applying the inequality $cl_{T',g'}(H) \leq cl_{T,g}(H)$ in the case where $(X, x_0) = (\mathbb{T}', t'_0)$ and $H = T'$ completes the proof. \square

Remark 2.4. If there exists a map $f : (\mathbb{T}, t_0) \rightarrow (\mathbb{T}', t'_0)$ such that $f_{\#}(T) \leq T'$ and $f_{\#}(g) = g'$, then we have $g' = f_{\#}(g) \in f_{\#}(cl_{T,g}(T)) \leq cl_{T,g}(f_{\#}(T)) \leq cl_{T,g}(T')$. Consequently, whenever such a map f exists, we may conclude that all three of the equivalent conditions in Proposition 2.3 hold.

The closure pairs of primary interest in this paper satisfy the following definition, which implies that the induced closure operator preserves conjugation.

Definition 2.5. A closure pair (T, g) for the test space (\mathbb{T}, t_0) is called *normal* if given any space (X, x_0) and subgroup $H \leq \pi_1(X, x_0)$, H is (T, g) -closed if and only if H^{α} is (T, g) -closed for every path $\alpha \in P(X, x_0)$.

Proposition 2.6. *If (T, g) is a normal closure pair and $N \leq \pi_1(X, x_0)$ is a normal subgroup, then $cl_{T,g}(N)$ is a normal subgroup of $\pi_1(X, x_0)$.*

Proof. Let $[\alpha] \in \pi_1(X, x_0)$. We check that $cl_{T,g}(N) \leq (cl_{T,g}(N))^\alpha$. Suppose $k \notin (cl_{T,g}(N))^\alpha$. Since $[\alpha]k[\alpha^-] \notin cl_{T,g}(N)$, there is a (T, g) -closed subgroup $H \leq \pi_1(X, x_0)$ such that $N \leq H$ and $[\alpha]k[\alpha^-] \notin H$. Since (T, g) is normal, H^α is (T, g) -closed. But since $k \notin H^\alpha$ and $N = N^\alpha \leq H^\alpha$, it follows that $k \notin cl_{T,g}(N)$. \square

Corollary 2.7. *Let (T, g) and (T', g') be normal closure pairs for (\mathbb{T}, t_0) and (\mathbb{T}', t'_0) respectively and let X be a space. Suppose that for every normal subgroup $N \leq \pi_1(X, x_0)$, N is (T, g) -closed if and only if N is (T', g') -closed. Then the closure operators $cl_{T,g}$ and $cl_{T',g'}$ agree on the normal subgroups of $\pi_1(X, x_0)$.*

Proof. If N is a normal subgroup, then $cl_{T,g}(N)$ contains N and is normal by Proposition 2.6. By assumption, $cl_{T,g}(N)$ is (T', g') -closed and therefore $cl_{T',g'}(N) \leq cl_{T,g}(N)$. By switching (T, g) and (T', g') and applying the same argument, we see that $cl_{T,g}(N) \leq cl_{T',g'}(N)$. \square

Lemma 2.8. *If (\mathbb{T}, t_0) is a well-pointed space, i.e. if $\{t_0\} \rightarrow \mathbb{T}$ is a cofibration, then every closure pair (T, g) for (\mathbb{T}, t_0) is normal.*

Proof. Fix a space (X, x_0) and subgroup $H \leq \pi_1(X, x_0)$. Suppose H is (T, g) -closed and $\alpha \in P(X, x_0)$ is any path. It suffices to show that H^α is (T, g) -closed. Let $f : (\mathbb{T}, t_0) \rightarrow (X, \alpha(1))$ be a map such that $f_\#(T) \leq H^\alpha$. Since $(\mathbb{T}, \{t_0\})$ has the homotopy extension property, there is a homotopy $H : \mathbb{T} \times [0, 1] \rightarrow X$ such that $H(d, 0) = f(d)$ and $H(t_0, s) = \alpha^-(s)$. Let $f_1 : (\mathbb{T}, t_0) \rightarrow (X, x_0)$ be the map $f_1(d) = H(d, 1)$. Then $(f_1)_\#([\gamma]) = [\alpha]f_\#([\gamma])[\alpha^-]$. It follows that $(f_1)_\#(T) = [\alpha]f_\#(T)[\alpha^-] \leq [\alpha]H^\alpha[\alpha^-] = H$. By assumption, $(f_1)_\#(g) \in H$, which implies $f_\#(g) \in H^\alpha$. \square

Remark 2.9. If (\mathbb{T}, t_0) is any space, we may add a “whisker” by forming the one-point union $\mathbb{T}^+ = (\mathbb{T}, t_0) \vee ([0, 1], 1)$. This new test space will be well-pointed if we take the basepoint t_0^+ to be the image of 0 in the union. If $j : \mathbb{T} \rightarrow \mathbb{T}^+$ and $\iota : [0, 1] \rightarrow \mathbb{T}^+$ are the inclusion maps and (T, g) is a closure pair for (\mathbb{T}, t_0) , then $([l]j_\#(T)[l^-], [l]j_\#(g)[l^-])$ is a normal closure pair for (\mathbb{T}^+, t_0^+) .

Remark 2.10. For a given space X and point $x_0 \in X$, we may assign to (X, x_0) the lattice of (T, g) -closed subgroups of $\pi_1(X, x_0)$. If (T, g) is a normal closure pair for (\mathbb{T}, t_0) , then this lattice is an invariant of the homotopy type of X . Indeed, if $h : X \rightarrow Y$ is a homotopy equivalence and $x_0 \in X$ is any point, then the induced isomorphism $h_\# : \pi_1(X, x_0) \rightarrow \pi_1(Y, f(x_0))$ satisfies the property that $H \leq \pi_1(X, x_0)$ is (T, g) -closed if and only if $h_\#(H)$ is (T, g) -closed.

Definition 2.11. A subgroup $H \leq \pi_1(X, x_0)$ is (T, g) -dense in $\pi_1(X, x_0)$ if $cl_{T,g}(H) = \pi_1(X, x_0)$.

Remark 2.12. It is apparently most practical to verify the following condition sufficient for density: if for every $k \in \pi_1(X, x_0)$, there is a map $f : (\mathbb{T}, t_0) \rightarrow (X, x_0)$ such that $f_\#(T) \leq H$ and $f_\#(g) = k$, then H is (T, g) -dense in $\pi_1(X, x_0)$.

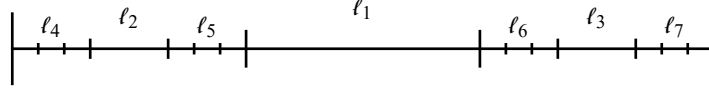
By applying Proposition 2.2, we obtain the following.

Corollary 2.13. *If $T \leq \pi_1(\mathbb{T}, t_0)$ is (T, g) -dense in $\pi_1(\mathbb{T}, t_0)$, then $H \leq \pi_1(X, x_0)$ is (T, g) -closed if and only if for every map $f : (\mathbb{T}, t_0) \rightarrow (X, x_0)$ such that $f_\#(T) \leq H$, we have $f_\#(\pi_1(\mathbb{T}, t_0)) \leq H$.*

3. THE HAWAIIAN EARRING AS A TEST SPACE

Let $C_n \subseteq \mathbb{R}^2$ be the circle of radius $\frac{1}{n}$ centered at $(\frac{1}{n}, 0)$ and $\mathbb{H} = \bigcup_{n \geq 1} C_n$ be the usual Hawaiian earring space with basepoint $b_0 = (0, 0)$. For $m \geq 1$, let $\mathbb{H}_{\geq m} = \bigcup_{n \geq m} C_n$ denote the smaller copies of the Hawaiian earring, all of which are homeomorphic to \mathbb{H} . We define some important loops in \mathbb{H} as follows:

- (1) Let ℓ_n define the canonical counterclockwise loop traversing the circle C_n . These loops generate the free subgroup $F = \langle [\ell_n] | n \geq 1 \rangle \leq \pi_1(\mathbb{H}, b_0)$.
- (2) Let $\ell_{\geq m}$ denote the “infinite concatenation” which is defined as ℓ_{n+m-1} on the interval $[\frac{n-1}{n}, \frac{n}{n+1}]$ and $\ell_{\geq m}(1) = b_0$. As a special case, we denote $\ell_{\infty} = \ell_{\geq 1}$.
- (3) Let $\mathcal{C} \subseteq [0, 1]$ be the standard middle third Cantor set. Write $[0, 1] \setminus \mathcal{C} = \bigcup_{n \geq 1} \bigcup_{k=1}^{2^{n-1}} I_n^k$ where I_n^k is an open interval of length $\frac{1}{3^n}$ and, for fixed n , the sets I_n^k are indexed by their natural ordering in $[0, 1]$. Let $\ell_{\tau} : [0, 1] \rightarrow \mathbb{H}$ be the “transfinite concatenation” defined so that $\ell_{\tau}(\mathcal{C}) = b_0$ and $\ell_{\tau} := \ell_{2^{n-1}+k-1}$ on I_n^k (see Figure 1).

FIGURE 1. The loop ℓ_{τ}

The fundamental group $\pi_1(\mathbb{H}, b_0)$ is uncountable and not free. However, it is naturally isomorphic to a subgroup of an inverse limit of free groups. Let $\mathbb{H}_{\leq n} = \bigcup_{m=1}^n C_m$ so that $\pi_1(\mathbb{H}_{\leq n}, b_0) = F_n$ is the group freely generated by the elements $[\ell_1], \dots, [\ell_n]$. For $n' > n$, the retractions $r_{n', n} : \mathbb{H}_{\leq n'} \rightarrow \mathbb{H}_{\leq n}$ collapsing $\bigcup_{n < m \leq n'} C_m$ to b_0 induce an inverse sequence

$$\cdots \rightarrow F_{n+1} \rightarrow F_n \rightarrow \cdots \rightarrow F_2 \rightarrow F_1$$

on fundamental groups, in which $F_{n+1} \rightarrow F_n$ deletes the letter $[\ell_n]$ from a given word. The inverse limit $\tilde{\pi}_1(\mathbb{H}, b_0) = \varprojlim_n F_n$ is the first shape homotopy group. The retractions $r_n : \mathbb{H} \rightarrow \mathbb{H}_{\leq n}$, which collapse $\mathbb{H}_{\geq n+1}$ to b_0 , induce a canonical homomorphism

$$\psi : \pi_1(\mathbb{H}, b_0) \rightarrow \tilde{\pi}_1(\mathbb{H}, b_0) \text{ where } \psi([\alpha]) = ((r_1)_{\#}([\alpha]), (r_2)_{\#}([\alpha]), \dots).$$

Since \mathbb{H} a one-dimensional planar Peano continuum, ψ is injective [17, 23]. Thus a homotopy class $[\alpha]$ is trivial if and only if for every $n \in \mathbb{N}$, the projection $(r_n)_{\#}([\alpha])$ as a word in the letters $[\ell_1], \dots, [\ell_n]$ reduces to the trivial word in F_n . Based on the injectivity of ψ , we also note that for every $n \in \mathbb{N}$, $\pi_1(\mathbb{H}, b_0)$ may be written as the free product $\pi_1(\mathbb{H}_{\leq n}, b_0) * \pi_1(\mathbb{H}_{\geq n+1}, b_0)$.

Example 3.1. Consider the closure pair $(F, [\ell_{\infty}])$ for the Hawaiian earring \mathbb{H} . Since $[\ell_{\infty}] \notin F$, the subgroup F is not $(F, [\ell_{\infty}])$ -closed. On the other hand, if a space X is semilocally simply connected at x_0 , then for every map $f : (\mathbb{H}, b_0) \rightarrow (X, x_0)$, there is an $m \geq 1$ such that $f_{\#}(\pi_1(\mathbb{H}_{\geq m}, b_0)) = 1$. Since every non-trivial element of $\pi_1(\mathbb{H}, b_0)$ may be written as a finite product of elements of $\pi_1(\mathbb{H}_{\geq m}, b_0)$ and the

free group $\langle [\ell_1], \dots, [\ell_{m-1}] \rangle$, it is clear that every subgroup of $\pi_1(X, x_0)$ is $(F, [\ell_\infty])$ -closed. For example, if (\mathbb{H}^+, b_0^+) is the space obtained by attaching a whisker as in Remark 2.9 and $j : \mathbb{H} \rightarrow \mathbb{H}^+$ and $\iota : [0, 1] \rightarrow \mathbb{H}^+$ are the inclusions, then $[\iota]j_\#(F)[\iota^-]$ is $(F, [\ell_\infty])$ -closed. However, the map $h : \mathbb{H}^+ \rightarrow \mathbb{H}$ collapsing the attached whisker to b_0 is a homotopy equivalence satisfying $h_\#([\iota]j_\#(F)[\iota^-]) = F$. According to Remark 2.10, $(F, [\ell_\infty])$ cannot be a normal closure pair.

Definition 3.2. An infinite sequence of paths $\alpha_n = \alpha_1, \alpha_2, \dots$ such that $\alpha_n(1) = \alpha_{n+1}(0)$ for each $n \geq 1$ is *null at* $x \in X$ if for every neighborhood U of x , there is an N such that $\alpha_n([0, 1]) \subseteq U$ for all $n \geq N$. The *infinite concatenation* of such a null-sequence is the path $\prod_{n=1}^\infty \alpha_j$ whose restriction to $\left[\frac{n-1}{n}, \frac{n}{n+1}\right]$ is α_n and $\alpha(1) = x$.

Note that a sequence $\alpha_n : ([0, 1], \{0, 1\}) \rightarrow (X, x)$ of loops is null at x if and only if there is a map $f : (\mathbb{H}, b_0) \rightarrow (X, x)$ such that $f \circ \ell_n = \alpha_n$, in which case $f \circ \ell_{\geq m} = \prod_{n=m}^\infty \alpha_n$.

Definition 3.3. (Homotopically Hausdorff relative to a subgroup [24]) We call X *homotopically Hausdorff relative to a subgroup* $H \leq \pi_1(X, x_0)$ if for every $x \in X$ and every path $\alpha : [0, 1] \rightarrow X$ from $\alpha(0) = x_0$ to $\alpha(1) = x$, only the trivial right coset of $H^\alpha = [\alpha^-]H[\alpha]$ in $\pi_1(X, x)$ has arbitrarily small representatives, that is, if for every $g \in \pi_1(X, x) \setminus H^\alpha$, there is an open set $U \in \mathcal{T}_x$ such that there is no loop $\delta : ([0, 1], \{0, 1\}) \rightarrow (U, x)$ with $H^\alpha g = H^\alpha[\delta]$.

The space X is *homotopically Hausdorff* if it is homotopically Hausdorff relative to the trivial subgroup $H = 1$.

Remark 3.4. Put differently, X is homotopically Hausdorff relative to H if and only if for every $x \in X$ and every path α from x_0 to x , we have

$$\bigcap_{U \in \mathcal{T}_x} H\pi(\alpha, U) = H.$$

To characterize the homotopically Hausdorff property, we apply the construction in Remark 2.9.

Definition 3.5. Let $\mathbb{H}^+ = \mathbb{H} \cup ([-1, 0] \times \{0\})$. We take $b_0^+ = (-1, 0)$ to be the basepoint of \mathbb{H}^+ so that (\mathbb{H}^+, b_0^+) is well-pointed. If $\iota : [0, 1] \rightarrow \mathbb{H}^+$ is the path $\iota(t) = (t - 1, 0)$, let

- (1) $c_n = [\iota \cdot \ell_n \cdot \iota^-]$,
- (2) $C = \langle c_n | n \geq 1 \rangle \leq \pi_1(\mathbb{H}^+, b_0^+)$,
- (3) $c_\infty = [\iota \cdot \ell_\infty \cdot \iota^-]$,
- (4) and $c_\tau = [\iota \cdot \ell_\tau \cdot \iota^-]$.

Theorem 3.6. If X is homotopically Hausdorff relative to $H \leq \pi_1(\mathbb{H}, b_0)$, then H is (C, c_∞) -closed. The converse holds if X is first countable.

Proof. Suppose $H \leq \pi_1(X, x_0)$ is not (C, c_∞) -closed and recall the characterization of the homotopically Hausdorff property from Remark 3.4. Then there exists a map $f : (\mathbb{H}^+, b_0^+) \rightarrow (X, x_0)$ such that $f_\#(C) \leq H$ and $f_\#(c_\infty) \notin H$. Set $x = f(b_0)$ and $\alpha = f \circ \iota$. Choose any $U \in \mathcal{T}_x$. By the continuity of f , there exists $m \geq 2$ such that the image of $f \circ \ell_{\geq m}$ lies in U . Since $f_\#(c_n) \in H$ for all $n \geq 1$, we have $f_\#(c_\infty) = f_\#(c_1 c_2 \cdots c_{m-1})[\alpha \cdot (f \circ \ell_{\geq m}) \cdot \alpha^-] \in H\pi(\alpha, U)$. Therefore, X cannot be homotopically Hausdorff relative to H .

For the converse, suppose X is first countable and is not homotopically Hausdorff relative to H . Then there exists a path α from x_0 to x and element $g \in (\bigcap_{U \in \mathcal{T}_x} H\pi(\alpha, U)) \setminus H$. Let $U_1 \supseteq U_2 \supseteq \dots$ be a countable neighborhood base at $\alpha(1)$. For each $n \geq 1$, find a loop $\delta_n \in \Omega(U_n, \alpha(1))$ such that $g \in H[\alpha \cdot \delta_n \cdot \alpha^-]$. Since $g \notin H$, observe that $[\alpha \cdot \delta_n \cdot \alpha^-] \notin H$ for each $n \geq 1$. Define a map $f : (\mathbb{H}^+, b_0^+) \rightarrow (X, x_0)$ so that $f \circ \iota = \alpha$ and $f \circ \ell_n = \delta_n \cdot \delta_{n+1}^-$. Notice $f_{\#}(c_n) = [\alpha \cdot \delta_n \cdot \alpha^-][\alpha \cdot \delta_{n+1} \cdot \alpha^-]^{-1} \in Hgg^{-1}H = H$ for all $n \geq 1$. Therefore $f_{\#}(C) \leq H$. On the other hand, since the infinite concatenation $\prod_{n \geq 2} (\delta_n^- \cdot \delta_n)$ is null-homotopic,

$$f_{\#}(c_{\infty}) = [\alpha \cdot \delta_1 \cdot \alpha^-][\alpha] \left[\prod_{n \geq 2} (\delta_n^- \cdot \delta_n) \right] [\alpha^-] = [\alpha \cdot \delta_1 \cdot \alpha^-] \notin H.$$

We conclude that H is not (C, c_{∞}) -closed. \square

Proposition 3.7. *C is (C, c_{τ}) -dense in $\pi_1(\mathbb{H}^+, b_0^+)$.*

Proof. We verify the property stated in Remark 2.12, which is sufficient for density. If \mathcal{C} is the middle third cantor set, let I_m be the unique component of $[0, 1] \setminus \mathcal{C}$ on which ℓ_{τ} is defined as the path ℓ_m . An arbitrary element $h \in \pi_1(\mathbb{H}^+, b_0^+)$ may be represented by a loop of the form $\iota \cdot \alpha \cdot \iota^-$ where $\alpha : [0, 1] \rightarrow \mathbb{H}$ is a loop based at b_0 . Let J_k , $k \in \mathbb{N}$ be the components of $[0, 1] \setminus \alpha^{-1}(b_0)$, of which we may assume there are infinitely many, and choose a function $\phi : \mathbb{N} \rightarrow \mathbb{N}$ such that the collection $\{J_k | k \in \mathbb{N}\}$ has the same order type as $\{I_{\phi(k)} | k \in \mathbb{N}\}$. For each k , there is an n_k , such that α restricted to $\overline{J_k}$ is homotopic within C_{n_k} (rel. endpoints) to either ℓ_{n_k} , $\ell_{n_k}^-$, or a constant. Accordingly, define $\beta : [0, 1] \rightarrow \mathbb{H}$ on $\overline{I_{\phi(k)}}$ as ℓ_{n_k} , $\ell_{n_k}^-$, or constant, respectively, and equal to b_0 elsewhere. Then $[\beta] = [\alpha] \in \pi_1(\mathbb{H}, b_0)$. Define a map $f : \mathbb{H}^+ \rightarrow \mathbb{H}^+$ so that $f \circ \iota = \iota$ and $f \circ \ell_n \equiv \beta|_{\overline{I_n}}$ for all n . Then $f_{\#}(C) \leq C$ and $f_{\#}(c_{\tau}) = [\iota \cdot \beta \cdot \iota^-] = h$. \square

Proposition 3.8. *If a subgroup $H \leq \pi_1(X, x_0)$ is (C, c_{τ}) -closed, then H is (C, c_{∞}) -closed. The converse holds if H is a normal subgroup of $\pi_1(X, x_0)$. In particular, the closure operators $cl_{C, c_{\infty}}$ and $cl_{C, c_{\tau}}$ agree on normal subgroups.*

Proof. Since C is (C, c_{τ}) -dense, we have $c_{\infty} \in \pi_1(\mathbb{H}^+, b_0^+) = cl_{C, c_{\tau}}(C)$. Apply Proposition 2.3. For the partial converse, suppose N is a normal, (C, c_{∞}) -closed subgroup of $\pi_1(X, x_0)$ and $f : \mathbb{H}^+ \rightarrow X$ is a map such that $f_{\#}(C) \leq N$. Recall Corollary 2.13. To obtain a contradiction, suppose there is a loop γ in \mathbb{H}^+ such that $f_{\#}([\gamma]) \notin N$. Set $\alpha = f \circ \iota$. We may assume $\gamma = \iota \cdot \beta \cdot \iota^-$ where β is a reduced loop in \mathbb{H} . Notice that for each $n \geq 1$, β is homotopic to a loop of the form $a_1 \cdot b_1 \cdot a_2 \cdot b_2 \cdots a_m \cdot b_m$ where a_j is constant or is in $\{\ell_1^{\pm}, \dots, \ell_{n-1}^{\pm}\}$ and b_j is a (possibly constant) loop with image in $\mathbb{H}_{\geq n}$. Therefore $f_{\#}([\gamma]) = f_{\#} \left(\prod_{j=1}^m [\iota \cdot a_j \cdot \iota^-][\iota \cdot b_j \cdot \iota^-] \right)$ is an element of

$$\prod_{j=1}^m f_{\#}(C) f_{\#}([\iota \cdot b_j \cdot \iota^-]) \leq \prod_{j=1}^m N f_{\#}([\iota \cdot b_j \cdot \iota^-])$$

and since N is normal,

$$f_{\#}([\gamma]) \in \prod_{j=1}^m N f_{\#}([\iota \cdot b_j \cdot \iota^-]) = N[\alpha \cdot \delta_n \cdot \alpha^-]$$

where $\delta_n = f \circ \prod_{j=1}^m b_j$ has image in $\mathbb{H}_{\geq n}$. Let $h : \mathbb{H}^+ \rightarrow X$ be the map defined by $h \circ \iota = \alpha$ and $h \circ \ell_n = \delta_n \cdot \delta_{n+1}^-$. Notice that

$$h_{\#}(c_n) = [\alpha \cdot \delta_n \cdot \alpha^-][\alpha \cdot \delta_{n+1}^- \cdot \alpha^-] \in N f_{\#}([\gamma]) f_{\#}([\gamma])^{-1} N = N.$$

Since $h_{\#}(C) \leq N$ and N is (C, c_{∞}) -closed, we have $h_{\#}(c_{\infty}) \in N$. But

$$h_{\#}(c_{\infty}) = [\alpha] \left[\prod_{n=1}^{\infty} \delta_n \cdot \delta_{n+1}^- \right] [\alpha^-] = [\alpha] [\delta_1] [\alpha^-] \in N,$$

which contradicts the fact that $[\alpha] [\delta_1] [\alpha^-] \in N f_{\#}([\gamma])$ and $f_{\#}([\gamma]) \notin N$.

The final statement of the proposition follows from Corollary 2.7. \square

Let $\mathbb{H}\mathbb{A}$ be the Harmonic Archipelago constructed by attaching 2-cells to \mathbb{H} along the loops $\ell_n \cdot \ell_{n+1}^-$ [2].

Corollary 3.9. *If X is first countable, then the following are equivalent:*

- (1) X is homotopically Hausdorff,
- (2) every map $f : \mathbb{H} \rightarrow X$ such that $f_{\#}(F) = 1 \leq \pi_1(X, f(b_0))$ induces the trivial homomorphism on π_1 ,
- (3) every map $f : \mathbb{H}\mathbb{A} \rightarrow X$ induces the trivial homomorphism on π_1 .

Proof. (1) \Leftrightarrow (2) follows from Corollary 2.13 and Propositions 3.7 and 3.8. For (2) \Rightarrow (3), observe that the inclusion $\mathbb{H} \rightarrow \mathbb{H}\mathbb{A}$ induces a surjection on π_1 . For (3) \Rightarrow (2), notice that $f : \mathbb{H} \rightarrow X$ extends to $f : \mathbb{H}\mathbb{A} \rightarrow X$ if $f_{\#}(F) = 1$. \square

Example 3.10. (A closure of a commutator subgroup) Set $G = \pi_1(\mathbb{H}, b_0)$ and let $[G, G]$ denote the commutator subgroup of G . Recall that the abelianization $G/[G, G]$ is isomorphic to the first singular homology group $H_1(\mathbb{H})$. Observe that $[G, G]$ is not (C, c_{∞}) -closed since infinite products of commutators such as $\left[\prod_{n=1}^{\infty} (\ell_{2n-1} \cdot \ell_{2n} \cdot \ell_{2n-1}^- \cdot \ell_{2n}^-) \right]$ are not elements of $[G, G]$; see Lemma 3.6 of [18].

Consider the infinite torus $T = \prod_{n \geq 1} C_n$ and the canonical embedding $m : \mathbb{H} \rightarrow T$ induced by the retractions $R_n : \mathbb{H} \rightarrow C_n$. Since each C_n is homotopically Hausdorff, $K = \ker(m_{\#}) = \bigcap_{n \geq 1} \ker((R_n)_{\#})$ is (C, c_{∞}) -closed. Since $\pi_1(T, x_0) \cong \prod_{n \geq 1} \mathbb{Z}$ is abelian, $[G, G] \leq K$ and thus $cl_{C, c_{\infty}}([G, G]) \leq K$. Suppose α is a loop in $\mathbb{H}_{\geq n}$ such that $[\alpha] \in K$. Then we may write $[\alpha] = \prod_{i=1}^m ([\delta_i][\ell_n]^{\epsilon_i})$ where the loop δ_i has image in $\mathbb{H}_{\geq n+1}$ and $\sum_{i=1}^m \epsilon_i = 0$. Since α and $\beta = \prod_{i=1}^m \delta_i$ are homologous in $\mathbb{H}_{\geq n}$, there is a loop γ in $\mathbb{H}_{\geq n}$ such that $[\gamma] \in [G, G]$ and $[\alpha] = [\gamma][\beta]$. Note $[\beta] \in K$. Thus given any element $[\alpha] \in K$, we may inductively construct loops γ_n in $\mathbb{H}_{\geq n}$ and β_n in $\mathbb{H}_{\geq n+1}$ such that $[\gamma_n] \in [G, G]$, $[\beta_n] \in K$, and $[\alpha] = (\prod_{i=1}^n [\gamma_i]) [\beta_n]$. By composing this sequence of shrinking homotopies, we see that $[\alpha] = \left[\prod_{n \geq 1} \gamma_n \right]$ and thus $[\alpha]$ is an infinite product of elements of $[G, G]$. We conclude that $cl_{C, c_{\infty}}([G, G]) = K$. See also Corollary 7.15.

In light of Remark 4.6 below, this computation sharpens the result in [11] that K is the topological closure of $[G, G]$ in G when G is equipped with its natural quotient topology.

Example 3.11. (Countable cut-points) For any closed subset A of X and point $x_0 \in X$, define $CCP(X, A, x_0)$ to be the subgroup

$$\{[\alpha] \in \pi_1(X, x_0) \mid \alpha^{-1}(A) \text{ is countable or } \alpha \text{ is constant}\}$$

of $\pi_1(X, x_0)$. Note that $CCP(X, \emptyset, x_0) = \pi_1(X, x_0)$ and $CCP(X, X, x_0) = 1$. It is well-known that a closed subspace B of \mathbb{R} is countable if and only if B is scattered in the sense that every nonempty subspace of B contains an isolated point. Hence $CCP(X, A, x_0)$ may also be described as the subgroup $\{[\alpha] \in \pi_1(X, x_0) \mid \alpha^{-1}(A) \text{ is scattered or } \alpha \text{ is constant}\}$.

The special case of $CCP(\mathbb{H}, \{b_0\}, b_0) \leq \pi_1(\mathbb{H}, b_0)$ has been studied in a variety of contexts. In [2], it is described as the subgroup of “countable order types.” By applying the results in Section 5 of [8], we may also identify $CCP(\mathbb{H}, \{b_0\}, b_0)$ with the group $Scatter(\aleph_0)$ of [8] and the group Sc of [14]. In both papers, $CCP(\mathbb{H}, \{b_0\}, b_0)$ is shown to be isomorphic to an uncountable free group.

Proposition 3.12. *If X is a one-dimensional metric space and $A \subseteq X$ is closed, then $CCP(X, A, x_0)$ is (C, c_∞) -closed.*

Proof. First, note that if $\alpha : [0, 1] \rightarrow X$ is a path such that $\alpha^{-1}(A)$ is countable, and δ is the reduced representative of α , then $\delta^{-1}(A)$ is also countable. Suppose $f : (\mathbb{H}^+, b_0^+) \rightarrow (X, x_0)$ is a map such that $f_\#(c_n) \in CCP(X, A, x_0)$ for all $n \in \mathbb{N}$. Set $\alpha = f \circ \iota$, $\gamma_n = f \circ \ell_n$, and $\gamma = \prod_{n=1}^\infty \gamma_n$. Since X is one-dimensional, we may assume the path α and each loop γ_n is either reduced or constant. If all γ_n are constant, then $f_\#(c_\infty) = [\alpha \cdot \gamma \cdot \alpha^-] = 1$. On the other hand, if $n_1 < n_2 < \dots$ is the entire sequence of indices such that γ_{n_k} is nonconstant, then by reducing constant subpaths, we have $[\gamma] = [\prod_{k=1}^\infty \gamma_{n_k}]$. Since we seek to show $f_\#(c_\infty) = [\alpha \cdot \gamma \cdot \alpha^-] \in CCP(X, A, x_0)$, we may assume that γ_n is reduced and nonconstant for every n .

If α is constant, then $f_\#(c_n) = [\gamma_n] \in CCP(X, A, x_0)$, which implies that $\gamma_n^{-1}(A)$ is countable for each $n \in \mathbb{N}$. By construction of γ , it follows that $\gamma^{-1}(A)$ is countable and thus $f_\#(c_\infty) \in CCP(X, A, x_0)$. Finally, suppose α is not constant. In this case, since γ_n is a null-sequence, the diameter of the homotopy between $\alpha \cdot \gamma_n \cdot \alpha^-$ and its reduced representative η_n approaches 0 as $n \rightarrow \infty$. Therefore, if $\alpha^{-1}(A)$ is uncountable, then there is a large enough N and a $0 < t < 1$ such that $\alpha|_{[0, t]}^{-1}(A)$ is uncountable and the reduced initial segment $\alpha|_{[0, t]}$ survives in the reduction of $\alpha \cdot \gamma_N \cdot \alpha^-$; a contradiction of the assumption $[\alpha \cdot \gamma_n \cdot \alpha^-] \in CCP(X, A, x_0)$ for all $n \in \mathbb{N}$. Thus $\alpha^{-1}(A)$ is countable. Since $\eta_n^{-1}(A)$ is also countable, we see that the preimage of A under $\alpha^- \cdot \eta_n \cdot \alpha$ is countable. Since γ_n is the reduced representative of $\alpha^- \cdot \eta_n \cdot \alpha$, $\gamma_n^{-1}(A)$ is countable for each $n \in \mathbb{N}$. Therefore, the preimage of A under $\alpha \cdot \gamma \cdot \alpha^-$ is countable, proving that $f_\#(c_\infty) \in CCP(X, A, x_0)$. \square

The previous example illustrates a difference in closure operators: since $C \leq CCP(\mathbb{H}^+, \{b_0\}, b_0^+)$ and $CCP(\mathbb{H}^+, \{b_0\}, b_0^+)$ is (C, c_∞) -closed, we have $cl_{C, c_\infty}(C) \leq CCP(\mathbb{H}^+, \{b_0\}, b_0^+) \neq \pi_1(\mathbb{H}^+, b_0^+)$. On the other hand, $cl_{C, c_\tau}(C) = \pi_1(\mathbb{H}^+, b_0^+)$ since C is (C, c_τ) -dense in $\pi_1(\mathbb{H}^+, b_0^+)$.

Corollary 3.13. $c_\tau \notin cl_{C, c_\infty}(C)$.

Definition 3.14. We say a space X has *transfinite products relative to a subgroup* $H \leq \pi_1(X, x_0)$ provided that for every pair of maps $a, b : (\mathbb{H}^+, b_0^+) \rightarrow (X, x_0)$ such that $a \circ \iota = b \circ \iota$ and $Ha_\#(c_n) = Hb_\#(c_n)$ for all $n \in \mathbb{N}$, we have $Ha_\#(g) = Hb_\#(g)$ for all $g \in \pi_1(\mathbb{H}^+, b_0^+)$. We say X has *transfinite products* if X has transfinite products relative to $H = 1$.

Remark 3.15. We briefly justify the terminology. A *transfinite word* over an alphabet A is a function $w : I \rightarrow A \cup A^{-1}$ defined on a linearly ordered domain I

such that $w^{-1}(s)$ is finite for every $a \in A \cup A^{-1}$ (note that if A is countable, then so is I). Given a transfinite word $w : I \rightarrow \{\ell_1^{\pm 1}, \ell_2^{\pm 1}, \ell_3^{\pm 1}, \dots\}$, choose components $((a_i, b_i))_{i \in I}$ of the complement $[0, 1] \setminus \mathcal{C}$ of the standard middle third cantor set \mathcal{C} such that $b_i < a_j$ for all $i < j$ and define a loop $\alpha_w : [0, 1] \rightarrow \mathbb{H}$ by $\alpha_w|_{[a_i, b_i]} \equiv w(i)$ and constant at b_0 elsewhere. Since $\pi_1(\mathbb{H}, b_0) \rightarrow \tilde{\pi}_1(\mathbb{H}, b_0)$ is injective, $[\alpha_w] \in \pi_1(\mathbb{H}, b_0)$ does not depend on the choice of the components.

Given a map $f : (\mathbb{H}, b_0) \rightarrow (X, x_0)$ and a transfinite word $w : I \rightarrow \{\ell_1^{\pm 1}, \ell_2^{\pm 1}, \ell_3^{\pm 1}, \dots\}$, we define $w_f = f_{\#}([\alpha_w]) \in \pi_1(X, x_0)$.

Let $(s_n)_{n \in \mathbb{N}}$ be a sequence in $\pi_1(X, x_0)$ such that $s_n = f_{\#}([\ell_n])$ for some map $f : (\mathbb{H}, b_0) \rightarrow (X, x_0)$ and let $w : I \rightarrow \{s_1^{\pm 1}, s_2^{\pm 1}, s_3^{\pm 1}, \dots\}$ be a transfinite word where $w(i) = s_{\nu(i)}^{\epsilon(i)}$. Put $\bar{w}(i) = \ell_{\nu(i)}^{\epsilon(i)}$. The space X has transfinite products if and only if the definition

$$\prod_{i \in I} s_{\nu(i)}^{\epsilon(i)} := \bar{w}_f$$

does not depend on f . This property allows for a well-defined notion of a subgroup transfinitely generated by a null-sequence of elements.

Definition 3.16. Let $P \leq \pi_1(\mathbb{H}^+, b_0^+)$ be the free subgroup generated by elements $p_n = [\iota][\ell_{2n-1} \cdot \ell_{2n}^-][\iota^-]$ for $n \geq 1$. Consider the maps $f_{\text{odd}}, f_{\text{even}} : \mathbb{H} \rightarrow \mathbb{H}$ satisfying $f_{\text{odd}} \circ \ell_n = \ell_{2n-1}$ and $f_{\text{even}} \circ \ell_n = \ell_{2n}$. Let $p_{\tau} = [\iota][(f_{\text{odd}} \circ \ell_{\tau}) \cdot (f_{\text{even}} \circ \ell_{\tau})^-][\iota^-]$ (see Figure 2).

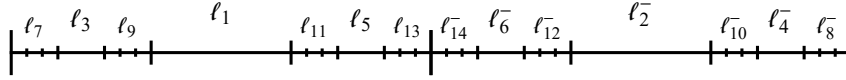


FIGURE 2. The loop $(f_{\text{odd}} \circ \ell_{\tau}) \cdot (f_{\text{even}} \circ \ell_{\tau})^-$

Proposition 3.17. *If $H \leq \pi_1(X, x_0)$ is (P, p_{τ}) -closed, then H is (C, c_{τ}) -closed.*

Proof. Consider the map $f : \mathbb{H}^+ \rightarrow \mathbb{H}^+$ defined so that $f \circ \iota = \iota$, $f \circ \ell_{2n-1} = \ell_n$, and $f \circ \ell_{2n}$ is constant. Since $f_{\#}(P) \leq C$ and $f_{\#}(p_{\tau}) = c_{\tau}$, we may apply Remark 2.4. \square

Proposition 3.18. *X has transfinite products relative to a subgroup $H \leq \pi_1(X, x_0)$ if and only if H is (P, p_{τ}) -closed.*

Proof. Suppose X does not have transfinite products relative to a subgroup $H \leq \pi_1(X, x_0)$. Then there are maps $a, b : (\mathbb{H}^+, b_0^+) \rightarrow (X, x)$ with $a \circ \iota = b \circ \iota$ and $a_{\#}(c_n)b_{\#}(c_n)^{-1} \in H$ for all $n \geq 1$ but with $a_{\#}(g)b_{\#}(g)^{-1} \notin H$ for some $g \in \pi_1(\mathbb{H}, b_0)$. Using the proof of Proposition 3.7, there is a map $k : \mathbb{H}^+ \rightarrow \mathbb{H}^+$ such that $k \circ \iota = \iota$, $k_{\#}(c_n)$ is either the identity or $c_{m_n}^{\epsilon_n}$ for some $m_n \geq 1$, $\epsilon_n \in \{\pm 1\}$ and $k_{\#}(c_{\tau}) = g$. Now $a \circ k$ and $b \circ k$ are maps $(\mathbb{H}^+, b_0^+) \rightarrow (X, x)$ such that either $(a \circ k)_{\#}(c_n)(b \circ k)_{\#}(c_n)^{-1} = 1$ or $(a \circ k)_{\#}(c_n)(b \circ k)_{\#}(c_n)^{-1} a_{\#}(c_{m_n}^{\epsilon_n})b_{\#}(c_{m_n}^{\epsilon_n})^{-1} \in H$. Additionally, $(a \circ k)_{\#}(c_{\tau})(b \circ k)_{\#}(c_{\tau})^{-1} = a_{\#}(g)b_{\#}(g)^{-1} \notin H$. Define a map $f : \mathbb{H}^+ \rightarrow X$ so that $f \circ \iota = a \circ k \circ \iota$, $f \circ \ell_{2n-1} = a \circ k \circ \ell_n$, and $f \circ \ell_{2n} = b \circ k \circ \ell_n$. Note $f_{\#}(p_n) = (a \circ k)_{\#}(c_n)(b \circ k)_{\#}(c_n)^{-1} \in H$ for each $n \geq 1$ and $f_{\#}(p_{\tau}) = (a \circ k)_{\#}(c_{\tau})(b \circ k)_{\#}(c_{\tau})^{-1} \notin H$. Thus H is not (P, p_{τ}) -closed.

For the converse, suppose H is not (P, p_τ) -closed. Then there is a map $f : \mathbb{H}^+ \rightarrow X$ such that $f_\#(P) \leq H$ and $f_\#(p_\tau) \notin H$. Define $a, b : \mathbb{H}^+ \rightarrow X$ such that $a \circ \iota = b \circ \iota = f \circ \iota$ and $a \circ \ell_n = f \circ \ell_{2n-1}$ and $b \circ \ell_n = f \circ \ell_{2n}$. Then $a_\#(c_n)b_\#(c_n)^{-1} = f_\#(p_n) \in H$ and $a_\#(c_\tau)b_\#(c_\tau)^{-1} = f_\#(p_\tau) \notin H$, which shows that X does not have transfinite products relative to H . \square

Proposition 3.19. *Suppose $N \leq \pi_1(X, x_0)$ is a subgroup containing the commutator subgroup of $\pi_1(X, x_0)$. Then the following are equivalent:*

- (1) N is (C, c_∞) -closed,
- (2) N is (C, c_τ) -closed,
- (3) N is (P, p_τ) -closed.

Proof. (3) \Rightarrow (2) and (2) \Rightarrow (1) follow from Propositions 3.17 and 3.8 respectively. To complete the proof, we show (1) \Rightarrow (3). Suppose N is (C, c_∞) -closed. Let $f : (\mathbb{H}^+, b_0^+) \rightarrow (X, x_0)$ be a map such that $f_\#(P) \leq N$. Recall the definition of p_τ and set $g = f_\#(p_\tau)$. To obtain a contradiction, suppose $g \notin N$. Since N contains the commutator subgroup, N is a normal subgroup of $\pi_1(X, x_0)$ whose factor group $\pi_1(X, x_0)/N$ is abelian. Thus, for each $n \geq 1$ there is a loop δ_n in $\mathbb{H}_{\geq 2n+1}$ such that the coset gN factors as:

$$\begin{aligned} gN &= \left(\prod_{k=1}^n (f_\#(c_{2k-1})N) (f_\#(c_{2k}^{-1})N) \right) f_\#([\iota \cdot \delta_n \cdot \iota^-])N \\ &= \left(\prod_{k=1}^n f_\#(p_k) \right) N f_\#([\iota \cdot \delta_n \cdot \iota^-])N \\ &= f_\#([\iota \cdot \delta_n \cdot \iota^-])N \end{aligned}$$

Thus $f_\#([\iota \cdot \delta_n \cdot \iota^-]) \notin N$ for any $n \geq 1$. We proceed as in Proposition 3.8. Define a map $h : \mathbb{H}^+ \rightarrow X$ so that $h \circ \iota = f \circ \iota$ and $h \circ \ell_n = f \circ (\delta_n \cdot \delta_{n+1}^-)$. We have $h_\#(C) \leq N$ and $h_\#(c_\infty) = f_\#([\iota \cdot \delta_1 \cdot \iota^-]) \notin N$; a contradiction of the assumption that N is (C, c_∞) -closed. \square

We complete this section by proving that $p_\tau \notin cl_{C, c_\tau}(P)$. Combining this fact with Proposition 2.3, we confirm that the closure operators cl_{C, c_τ} and cl_{P, p_τ} are not equal.

Lemma 3.20. *Consider the map $f : \mathbb{H} \rightarrow \mathbb{H}$ satisfying $f \circ \ell_n = \ell_{2n-1} \cdot \ell_{2n}^-$. Then*

- (1) $f_\# : \pi_1(\mathbb{H}, b_0) \rightarrow \pi_1(\mathbb{H}, b_0)$ is injective,
- (2) a based loop α in \mathbb{H} is reduced if and only if $f \circ \alpha$ is reduced in \mathbb{H} ,
- (3) a sequence α_k of based reduced loops in \mathbb{H} is null if and only if $f \circ \alpha_k$ is null,
- (4) If α is a based reduced loop, and there is an increasing sequence $0 < t_1 < t_2 < \dots < 1$ converging to 1 such that $\alpha(t_k) = b_0$ and $[\alpha|_{[0, t_k]}] \in f_\#(\pi_1(\mathbb{H}, b_0))$ for each $k \geq 1$, then $[\alpha] \in f_\#(\pi_1(\mathbb{H}, b_0))$.

Proof. (1) Let α be a loop in \mathbb{H} such that $[\alpha] \neq 1$ in $\pi_1(\mathbb{H}, b_0)$. Then there is an $n \geq 1$, $g_1, \dots, g_k \in \pi_1(\mathbb{H}_{\geq n+1}, b_0)$, and $h_1, \dots, h_k \in \pi_1(\mathbb{H}_{\leq n}, b_0) = F_n$ such that $[\alpha] = g_1 h_1 g_2 h_2 \dots g_k h_k$ and $1 \neq h_1 h_2 \dots h_k \in F_n$. Thus

$$f_\#([\alpha]) = f_\#(g_1) f_\#(h_1) f_\#(g_2) f_\#(h_2) \dots f_\#(g_k) f_\#(h_k)$$

where $f_{\#}(g_i) \in \pi_1(\mathbb{H}_{\geq 2n+1}, b_0)$ and $f_{\#}(h_i) \in \pi_1(\mathbb{H}_{\leq 2n}, b_0) = F_{2n}$. But the restriction of f to $\mathbb{H}_{\leq n}$ induces an injection $F_n \rightarrow F_{2n}$ on π_1 . Thus

$$f_{\#}(h_1)f_{\#}(h_2) \cdots f_{\#}(h_k) = f_{\#}(h_1 h_2 \cdots h_k) \neq 1$$

in F_{2n} . It follows that $f_{\#}([\alpha]) \neq 1$.

(2) Clearly, if α is not reduced, then $f \circ \alpha$ is not reduced. For the converse, let α be reduced. Then $A = \alpha^{-1}(b_0)$ is nowhere dense and if (a, b) is a component of $[0, 1] \setminus A$, then $\alpha|_{[a, b]}$ is a loop of the form ℓ_n or ℓ_n^- . Note that $f \circ \alpha|_{[a, b]}$ must be a loop of the form $\ell_{2n-1} \cdot \ell_{2n}^-$ or $\ell_{2n} \cdot \ell_{2n-1}^-$. To obtain a contradiction, suppose $f \circ \alpha$ is not reduced. Then there are $0 \leq s < t \leq 1$ such that $f \circ \alpha|_{[s, t]}$ is a null-homotopic loop in \mathbb{H} . Given the definition of f and the fact that α is reduced, we may assume $f \circ \alpha|_{[s, t]}$ is based at b_0 .

If $\alpha|_{[s, t]}$ is a loop, then $[\alpha|_{[s, t]}] \neq 1$ since α is a reduced. However, $[f \circ \alpha|_{[s, t]}] = 1$, which contradicts (1).

If $\alpha(s) \neq b_0$, then by definition of f , we have $s = \frac{a+b}{2}$ for a component (a, b) of $[0, 1] \setminus A$. The path $f \circ \alpha|_{[a, b]}$ is either of the form $\ell_{2n-1} \cdot \ell_{2n}^-$ or $\ell_{2n} \cdot \ell_{2n-1}^-$. First, suppose $f \circ \alpha|_{[a, b]} \equiv \ell_{2n-1} \cdot \ell_{2n}^-$. Since $[f \circ \alpha|_{[s, t]}] = [\ell_{2n}^-][f \circ \alpha|_{[b, t]}] = 1$, we must have a positive, equal number of appearances of ℓ_{2n} and ℓ_{2n}^- as subloops of $f \circ \alpha|_{[s, t]}$. Note that ℓ_{2n} and ℓ_{2n}^- cannot occur as consecutive subloops of $f \circ \alpha|_{[s, t]}$ since α is reduced. Since $\pi_1(\mathbb{H}, b_0)$ may be written as the free product $\pi_1(C_{2n}, b_0) * \pi_1(\bigcup_{m \neq 2n} C_m, b_0)$ and $f \circ \alpha|_{[s, t]}$ reduces completely in \mathbb{H} , it must have a subloop of the form $\ell_{2n} \cdot \beta \cdot \ell_{2n}^-$ or $\ell_{2n}^- \cdot \beta \cdot \ell_{2n}$ where β is a nonconstant, null-homotopic loop in $\bigcup_{m \neq 2n} C_m$. By definition of f , there are $s < s' < t' < t$ such that $\alpha(s') = \alpha(t') = b_0$ and $f \circ \alpha|_{[s', t']} \equiv \beta$. But now $\alpha|_{[s', t']}$ is a reduced loop such that $f_{\#}([\alpha|_{[s', t]}]) = 1$; a contradiction of (1). On the other hand, if $f \circ \alpha|_{[a, b]} \equiv \ell_{2n} \cdot \ell_{2n-1}^-$, we may apply a similar argument using the identification $\pi_1(\mathbb{H}, b_0) = \pi_1(C_{2n-1}, b_0) * \pi_1(\bigcup_{m \neq 2n-1} C_m, b_0)$.

If $\alpha(t) \neq b_0$, we may apply the argument from the previous paragraph.

(3) By continuity, $f \circ \alpha_k$ is null whenever α_k is null. If α_k is a sequence of reduced loops which is not null, then there is an n such that infinitely many of the loops α_k traverse at least one of the circles C_1, \dots, C_n . Consequently, infinitely many of the loops $f \circ \alpha_k$ traverse at least one of the circles C_1, \dots, C_{2n} . Thus $f \circ \alpha_k$ is not null.

(4) Since α is reduced, $\alpha|_{[0, t_k]}$ is reduced for each k . Moreover, since $[\alpha|_{[0, t_k]}] \in f_{\#}(\pi_1(\mathbb{H}, b_0))$, by (2) there is a unique, reduced loop $\eta_k : [0, t_k] \rightarrow \mathbb{H}$ such that $f \circ \eta_k = \alpha|_{[0, t_k]}$. By uniqueness, $\eta_{k+1}|_{[0, t_k]} = \eta_k$. Define $\eta : [0, 1] \rightarrow X$ such that $\eta(s) = \eta_k(s)$ for $0 \leq s \leq t_k$ and $\eta(1) = b_0$. By (3), η is a loop with $f \circ \eta \equiv \alpha$, so that $[\alpha] \in f_{\#}(\pi_1(\mathbb{H}, b_0))$. \square

Lemma 3.21. *If $f : \mathbb{H} \rightarrow \mathbb{H}$ is the map defined in Lemma 3.20, then $H = f_{\#}(\pi_1(\mathbb{H}, b_0))$ is (C, c_r) -closed.*

Proof. Let $g : (\mathbb{H}^+, b_0^+) \rightarrow (\mathbb{H}, b_0)$ be a map such that $g_{\#}(C) \leq H$. Since \mathbb{H} is locally contractible at all points of $\mathbb{H} \setminus \{b_0\}$, we may focus on the case when $g(b_0) = b_0$. We may also assume that $\alpha = g \circ \iota$ and $\gamma_n = g \circ \ell_n$, $n \in \mathbb{N}$ are reduced (or constant) loops satisfying $[\alpha \cdot \gamma_n \cdot \alpha^-] \in H$ and that each γ_n is nonconstant. We first prove that $[\alpha] \in H$. This is clear if α is constant. Suppose α is not constant.

Case I: Suppose there is a $0 < t < 1$ such that $\alpha|_{[t, 1]} \equiv \ell_m^\pm$ for some $m \in \mathbb{N}$. Find N such that γ_N has image in $\mathbb{H}_{\geq m+2}$. Then $\alpha \cdot \gamma_N \cdot \alpha^-$ is already reduced. Thus there is a unique loop β such that $f \circ \beta = \alpha \cdot \gamma_N \cdot \alpha^-$. It suffices to show $1/3 \in \beta^{-1}(b_0)$ since this would imply $[\alpha] = f_{\#}([\beta|_{[0, 1/3]}]) \in H$. Suppose $1/3 \notin \beta^{-1}(b_0)$. Since

$f \circ \beta(1/3) = b_0$, it follows from the definition of f that there is a component (r, s) of $[0, 1] \setminus \beta^{-1}(b_0)$ such that $1/3 = \frac{r+s}{2}$ and $f \circ \beta|_{[r,s]} \equiv \ell_{2k} \cdot \ell_{2k-1}^-$ or $f \circ \beta|_{[r,s]} \equiv \ell_{2k-1} \cdot \ell_{2k}^-$. But this is impossible since $f \circ \beta|_{[r,1/3]}$ has image in C_m and $f \circ \beta|_{[1/3,s]}$ has image in $\mathbb{H}_{\geq m+2}$.

Case II: Suppose there is an increasing sequence $0 < s_1 < s_2 < \dots < 1$ converging to 1 such that $\alpha|_{[s_k, 1]}$ is a non-trivial, reduced loop. Let $A_k = \{t \in (s_k, s_{k+1}) | \alpha(t) = b_0\}$; we may assume that $|A_k| \geq 1$ for each k . We show that there exists $t_k \in [s_k, s_{k+1}]$ such that $[\alpha|_{[0, t_k]}] \in H$. By (4) of Lemma 3.20, this is enough to show that $[\alpha] \in H$. Fix k and find an m such that ℓ_m^\pm appears in $\alpha|_{[s_{k+1}, s_{k+2}]}$. Find an N such that γ_N has image in $\mathbb{H}_{\geq m+1}$. Let δ be the reduced representative of $\alpha \cdot \gamma_N \cdot \alpha^-$. Then there is a $0 < q < 1$ such that $\delta|_{[0, q]} \equiv \alpha|_{[0, s_{k+1}]}$. Additionally, there is a unique $0 < p < q$ such that $\delta|_{[p, q]} \equiv \alpha|_{[s_k, s_{k+1}]}$. Since $[\delta] \in H$, there is a reduced loop β such that $f \circ \beta = \delta$. If $\beta(q) = b_0$, set $t_k = s_{k+1}$; it is clear that $[\alpha|_{[0, t_k]}] = [f \circ \beta|_{[0, q]}] \in H$. If $\beta(q) \neq b_0$, then $q = \frac{r+s}{2}$ for some component (r, s) of $[0, 1] \setminus \beta^{-1}(b_0)$. Since $|A_k| \geq 1$, we have $p < r < q$. Take t_k to be the unique value such that $\delta|_{[0, r]} \equiv \alpha|_{[0, t_k]}$. Since $\beta(r) = b_0$, we have $[\alpha|_{[0, t_k]}] = [f \circ \beta|_{[0, r]}] \in H$.

Cases I and II together prove that $[\alpha] \in H$. Since we also have $[\alpha \cdot \gamma_n \cdot \alpha^-] \in H$ for each $n \in \mathbb{N}$, we have $[\gamma_n] \in H$ for each $n \in \mathbb{N}$. By (2) of Lemma 3.20, $\alpha = f \circ \beta$ and $\gamma_n = f \circ \zeta_n$ for reduced loops β and ζ_n . Since γ_n is a null-sequence, ζ_n is a null-sequence by (3) of Lemma 3.20. Define $h : \mathbb{H}^+ \rightarrow \mathbb{H}$ by $h \circ \iota = \beta$ and $h \circ \ell_n = \zeta_n$. Since $f \circ h = g$, it follows that $g_\#(c_\tau) = f_\#(h_\#(c_\tau)) \in H$, completing the proof. \square

Theorem 3.22. $p_\tau \notin cl_{C, c_\tau}(P)$.

Proof. Let $f : \mathbb{H} \rightarrow \mathbb{H}$ be the map defined in Lemma 3.20 and $f^+ : (\mathbb{H}^+, b_0^+) \rightarrow (\mathbb{H}^+, b_0^+)$ be the map where $f \circ \iota = \iota$ and $f^+|_{\mathbb{H}} = f$. Let $K = f_\#^+(\pi_1(\mathbb{H}^+, b_0^+))$. We show that $cl_{C, c_\tau}(P) = K$ and $p_\tau \notin K$.

Note that $P = f_\#^+(C)$. Since C is (C, c_τ) -dense, we have $K = f_\#^+(cl_{C, c_\tau}(C)) \leq cl_{C, c_\tau}(f_\#^+(C)) = cl_{C, c_\tau}(P)$. If $h : \mathbb{H}^+ \rightarrow \mathbb{H}$ is the homotopy equivalence collapsing the attached whisker to b_0 , then $h_\#(K) = f_\#(\pi_1(\mathbb{H}, b_0))$. By Lemma 3.21, $f_\#(\pi_1(\mathbb{H}, b_0))$ is (C, c_τ) -closed. Applying Remark 2.10, we see that K is (C, c_τ) -closed. Finally, since $P = [\iota]f_\#(C)[\iota^-] \leq K$ and K is (C, c_τ) -closed, we have $cl_{C, c_\tau}(P) \leq K$. This proves $cl_{C, c_\tau}(P) = K$.

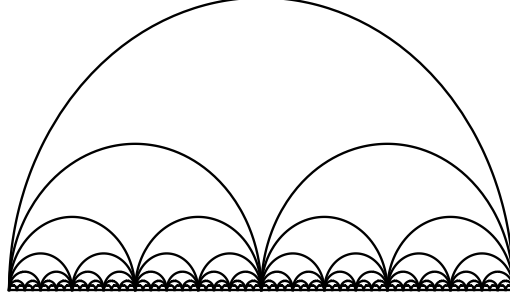
The only non-trivial elements of K that may be factored as a product $[\iota][\alpha][\beta][\iota^-]$ with α having image in $\bigcup_{n \text{ odd}} C_n$ and β having image in $\bigcup_{n \text{ even}} C_n$ are the elements $p_n \in P$. Since p_τ has such a factorization yet $p_\tau \notin P$, we conclude that $p_\tau \notin K$. \square

4. A DYADIC ARC SPACE

A pair (n, j) of integers is *dyadic unital* if $n \geq 1$ and $1 \leq j \leq 2^{n-1}$ or equivalently if $\frac{2j-1}{2^n} \in (0, 1)$. Let \mathcal{D} denote the set of dyadic unital pairs. For each dyadic unital pair (n, j) , let

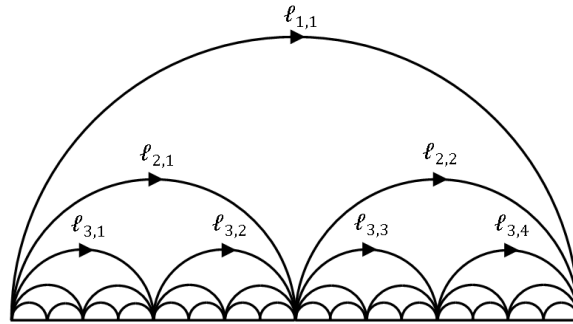
$$\mathbb{D}(n, j) = \left\{ (x, y) \in \mathbb{R}^2 \mid \left(x - \frac{2j-1}{2^n} \right)^2 + y^2 = \left(\frac{1}{2^n} \right)^2, x \geq 0 \right\}$$

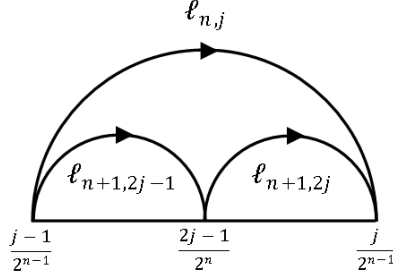
denote the upper semicircle of radius $\frac{1}{2^n}$ centered at $(\frac{2j-1}{2^n}, 0)$. The *base arc* is the interval $B = [0, 1] \times \{0\}$. We consider the union $\mathbb{D} = B \cup \bigcup_{(n, j) \in \mathcal{D}} \mathbb{D}(n, j)$ topologized as a subspace of \mathbb{R}^2 and with basepoint $d_0 = (0, 0)$ (see Figure 3). We call $\mathbb{D}(n) = \bigcup_{j=1}^{2^{n-1}} \mathbb{D}(n, j)$ the n -th level of \mathbb{D} .

FIGURE 3. The space \mathbb{D}

For each dyadic unital pair (n, j) , let $\ell_{n,j} : [0, 1] \rightarrow \mathbb{D}(n, j)$ be the arc $\ell_{n,j}(t) = \left(\frac{t+j-1}{2^{n-1}}, \frac{1}{2^{n-1}} \sqrt{t-t^2} \right)$ from $(\frac{j-1}{2^{n-1}}, 0)$ to $(\frac{j}{2^{n-1}}, 0)$ (see Figures 4 and 5). We say a path in \mathbb{D} is *standard* if it is of the form $\ell_{n,j}$ or $(\ell_{n,j})^-$. To simplify notation, we define:

- (1) $\lambda_{n,m} = \prod_{j=1}^m \ell_{n,j}$ to be the concatenation of standard paths on the n -th level from d_0 to $(\frac{m}{2^{n-1}}, 0)$. We allow $\lambda_{n,0}$ to denote the constant path at d_0 .
- (2) $\lambda_n = \lambda_{n,2^{n-1}} = \prod_{j=1}^{2^{n-1}} \ell_{n,j}$ to be the path from $(0,0)$ to $(1,0)$ on the n -th level.
- (3) $\lambda_\infty(t) = (t, 0)$ to be the unit speed path on the base arc.

FIGURE 4. The canonical arcs $\ell_{n,j} : [0, 1] \rightarrow \mathbb{D}$, $1 \leq n \leq 3$

FIGURE 5. The canonical arc $\ell_{n,j} : [0, 1] \rightarrow \mathbb{D}$

As with the Hawaiian earring, the fundamental group of \mathbb{D} can be understood as a subgroup of an inverse limit of free groups. Consider the finite graph $E_n = B \cup \bigcup_{m=1}^n \mathbb{D}(m)$ whose fundamental group $\pi_1(E_n, d_0) = F_{2^n-1}$ is free on $2^n - 1$ generators. For $n' > n$ the retractions $E_{n'} \rightarrow E_n$, which collapse a point $(s, t) \in \bigcup_{n < m \leq n'} \mathbb{D}(m)$ to the corresponding point $(s, 0)$ on the base arc, induce an inverse sequence on π_1 whose limit $\tilde{\pi}_1(\mathbb{D}, d_0) = \varprojlim_n F_{2^n-1}$ is the first shape homotopy group. The retractions $r_n : \mathbb{D} \rightarrow E_n$, which collapse $\bigcup_{m > n} \mathbb{D}(m)$ onto the base arc by vertical projection, induce a canonical homomorphism $\psi : \pi_1(\mathbb{D}, d_0) \rightarrow \tilde{\pi}_1(\mathbb{D}, d_0)$. Since \mathbb{D} is a one-dimensional planar Peano continuum, ψ is injective. Thus two loops $\alpha, \beta \in \Omega(\mathbb{D}, d_0)$ are homotopic if and only if for every $n \in \mathbb{N}$ the projections of α and β are homotopic in E_n .

4.1. The subgroup $S \leq \pi_1(\mathbb{D}, d_0)$ and the homotopically path Hausdorff property.

Definition 4.1. (Homotopically path Hausdorff relative to a subgroup) We call X *homotopically path Hausdorff relative to H* if for every pair of paths $\alpha, \beta \in P(X, x_0)$ such that $\alpha(1) = \beta(1)$ and $[\alpha \cdot \beta^-] \notin H$, there is an integer $n \geq 1$ and a sequence of open sets $U_1, U_2, \dots, U_{2^{n-1}}$ with $\alpha\left(\left[\frac{j-1}{2^{n-1}}, \frac{j}{2^{n-1}}\right]\right) \subseteq U_j$, such that if $\gamma : [0, 1] \rightarrow X$ is another path satisfying $\gamma\left(\left[\frac{j-1}{2^{n-1}}, \frac{j}{2^{n-1}}\right]\right) \subseteq U_j$ for $1 \leq j \leq 2^{n-1}$ and $\gamma\left(\frac{j}{2^{n-1}}\right) = \alpha\left(\frac{j}{2^{n-1}}\right)$ for $0 \leq j \leq 2^{n-1}$, then $[\gamma \cdot \beta^-] \notin H$.

We call X *homotopically path Hausdorff* if it is homotopically path Hausdorff relative to the trivial subgroup $H = 1$.

Remark 4.2. The original definition of the homotopically path Hausdorff property given in [22] does not use dyadic rationals, however, it is equivalent to the definition used here, which is more convenient for our purposes.

Definition 4.3. For $n \geq 1$, let $s_n = [\lambda_n \cdot (\lambda_{n+1})^-]$. Let S be the subgroup of $\pi_1(\mathbb{D}, d_0)$ freely generated by $\{s_n | n \in \mathbb{N}\}$ and $d_\infty = [\lambda_1 \cdot \lambda_\infty^-]$.

Although (\mathbb{D}, d_0) is not well-pointed, we use the self-similarity of \mathbb{D} to show that (S, d_∞) is a normal closure pair.

Proposition 4.4. (S, d_∞) is a normal closure pair for (\mathbb{D}, d_0) .

Proof. Let \mathbb{D}^+ be the well-pointed space constructed in Remark 2.9 with accompanying normal closure pair $(S', d'_\infty) = ([\iota]i_\#(S)[\iota^-], [\iota]i_\#(d_\infty)[\iota^-])$ where $i : \mathbb{D} \rightarrow \mathbb{D}^+$ and $\iota : [0, 1] \rightarrow \mathbb{D}^+$ are the inclusion maps. Identify $\mathbb{D}^+ = B \cup \bigcup_{n=2}^\infty \bigcup_{j=2^{n-2}+1}^{2^{n-1}} \mathbb{D}(n, j)$

and define a canonical retraction $r : (\mathbb{D}, d_0) \rightarrow (\mathbb{D}^+, d_0)$ collapsing $\mathbb{D} \setminus \mathbb{D}^+$ vertically onto the arc $([0, 1/2] \times \{0\}) \cup \mathbb{D}(2, 2)$. Since $r_\#(S) = S'$ and $r_\#(d_\infty) = d'_\infty$, we have $d'_\infty \in cl_{S, d_\infty}(S')$ by Remark 2.4. To prove the proposition, suppose $H \leq \pi_1(X, x_0)$ is (S, d_∞) -closed, $\alpha \in P(X, x_0)$ is a path, and $f : (\mathbb{D}, d_0) \rightarrow (X, \alpha(1))$ is a map such that $f_\#(S) \leq H^\alpha$. It suffices to show that $f_\#(d_\infty) \in H^\alpha$. The map f and the path α uniquely induce a map $k : (\mathbb{D}^+, d_0) \rightarrow (X, x_0)$ satisfying $k \circ \iota = \alpha$ and $k|_{\mathbb{D}} = f$. Since $k_\#(S') \leq H$, we have

$$[\alpha]f_\#(d_\infty)[\alpha^-] = k_\#(d'_\infty) \in k_\#(cl_{S, d_\infty}(S')) \leq cl_{S, d_\infty}(k_\#(S')) \leq cl_{S, d_\infty}(H) = H.$$

Thus $f_\#(d_\infty) \in H^\alpha$. \square

Theorem 4.5. *If X is homotopically path Hausdorff relative to H , then H is (S, d_∞) -closed. The converse holds if the path space $P(X)$ is first countable, for instance, if X is metrizable.*

Proof. If H is not (S, d_∞) -closed, then there is a map $f : (\mathbb{D}, d_0) \rightarrow (X, x_0)$ such that $f_\#(S) \subseteq H$ and $f_\#(d_\infty) \notin H$. Set $\alpha = f \circ \lambda_\infty$, $\beta = f \circ \lambda_1$, and $\alpha_n = f \circ \lambda_n$ for $n \geq 2$. Note that $\alpha_n \rightarrow \alpha$ in $P(X)$ and that $[\alpha \cdot \beta^-] \notin H$. Pick any $n \geq 1$ and sequence of neighborhoods $U_1, U_2, \dots, U_{2^{n-1}}$ such that $\alpha\left(\left[\frac{j-1}{2^{n-1}}, \frac{j}{2^{n-1}}\right]\right) \subseteq U_j$ for each j . Choose N so that $\alpha_N \in \bigcap_{j=1}^{2^{n-1}} \langle \left[\frac{j-1}{2^{n-1}}, \frac{j}{2^{n-1}}\right], U_j \rangle$. In particular,

$$\begin{aligned} \alpha_N\left(\left[\frac{j-1}{2^{n-1}}, \frac{j}{2^{n-1}}\right]\right) &\subseteq U_j \text{ for } 1 \leq j \leq 2^{n-1}, \\ \alpha_N\left(\frac{j}{2^{n-1}}\right) &= \alpha\left(\frac{j}{2^{n-1}}\right) \text{ for } 0 \leq j \leq 2^{n-1}. \end{aligned}$$

Put $\gamma = \alpha_N$. Since $[\alpha_n \cdot \alpha_{n+1}^-] = f_\#(s_n) \in H$ for each $n \geq 1$, we have $[\gamma \cdot \beta^-] = \left(\prod_{n=1}^{N-1} [\alpha_n \cdot \alpha_{n+1}^-]\right)^{-1} \in H$. Thus X cannot be homotopically path Hausdorff relative to H .

For the converse, suppose $P(X)$ is first countable. If X is not homotopically path Hausdorff relative to H , then there are paths $\alpha, \beta : [0, 1] \rightarrow X$, $\alpha(0) = \beta(0) = x_0$ and $\alpha(1) = \beta(1)$ with the property that for any integer $n \geq 1$ and sequence of open sets $U_1, U_2, \dots, U_{2^{n-1}}$ with $\alpha\left(\left[\frac{j-1}{2^{n-1}}, \frac{j}{2^{n-1}}\right]\right) \subseteq U_j$, then there is a path $\gamma : [0, 1] \rightarrow X$ satisfying

- (1) $\gamma\left(\left[\frac{j-1}{2^{n-1}}, \frac{j}{2^{n-1}}\right]\right) \subseteq U_j$ for $1 \leq j \leq 2^{n-1}$,
- (2) $\gamma\left(\frac{j}{2^{n-1}}\right) = \alpha\left(\frac{j}{2^{n-1}}\right)$ for $0 \leq j \leq 2^{n-1}$,
- (3) and $[\gamma \cdot \beta^-] \in H$.

Take a countable, nested neighborhood base $\mathcal{U}_1 \supset \mathcal{U}_2 \supset \dots$ at α . We may assume each neighborhood is of the form

$$\mathcal{U}_p = \bigcap_{j=1}^{2^{n(p)-1}} \left\langle \left[\frac{j-1}{2^{n(p)-1}}, \frac{j}{2^{n(p)-1}}\right], U_{n(p),j} \right\rangle$$

for some increasing sequence $1 = n(1) < n(2) < n(3) < \dots$ of natural numbers. By assumption, there is a path $\alpha_{n(p)} : [0, 1] \rightarrow X$ satisfying

- (1) $\alpha_{n(p)}\left(\left[\frac{j-1}{2^{n(p)-1}}, \frac{j}{2^{n(p)-1}}\right]\right) \subseteq U_{n(p),j}$ for $1 \leq j \leq 2^{n(p)-1}$,
- (2) $\alpha_{n(p)}\left(\frac{j}{2^{n(p)-1}}\right) = \alpha\left(\frac{j}{2^{n(p)-1}}\right)$ for $0 \leq j \leq 2^{n(p)-1}$,
- (3) $[\alpha_{n(p)} \cdot \beta^-] \in H$.

If $n(p) < n < n(p+1)$, set $\alpha_n = \alpha_{n(p+1)}$. We have $\alpha_n \rightarrow \alpha$ in $P(X)$ and for every $n \geq 1$, $\alpha_n\left(\frac{j}{2^{n-1}}\right) = \alpha\left(\frac{j}{2^{n-1}}\right)$ for $0 \leq j \leq 2^{n-1}$. Thus we obtain a unique map $f : (\mathbb{D}, d_0) \rightarrow (X, x_0)$ such that $f \circ \lambda_n = \alpha_n$ and $f \circ \lambda_\infty = \alpha$. For any

given n , we have $\alpha_n = \alpha_{n(p)}$ for some p and thus $[\alpha_n \cdot \beta^-] = [\alpha_{n(p)} \cdot \beta^-] \in H$. It follows that $f_\#(s_n) = [\alpha_n \cdot \alpha_{n+1}^-] = [\alpha_n \cdot \beta^-][\alpha_{n+1} \cdot \beta^-]^{-1} \in H$ for each n and therefore $f_\#(S) \leq H$. Moreover, $f_\#(d_\infty) = [\alpha_1 \cdot \alpha^-] = [\alpha_1 \cdot \beta^-][\alpha \cdot \beta^-]^{-1} \notin H$ since $[\alpha \cdot \beta^-] \notin H$. \square

Remark 4.6. The fundamental group $\pi_1(X, x_0)$ inherits a natural topology when it is viewed as the quotient space of $\Omega(X, x_0)$. Equipped with this topology, the fundamental group may fail to be a topological group, however, it is a quasitopological group in the sense that inversion is continuous and multiplication is continuous in each variable; for more on this topology see [6]. In a quasitopological group G , the topological closure \bar{H} of a subgroup $H \leq G$ is still a subgroup of G [1]. Additionally, for a locally path-connected space X , the property of being homotopically path Hausdorff relative to H is equivalent to H being closed in G [6, Lemma 9]. Combining these facts with Theorem 4.5, it follows that if X is a locally path-connected metric space, then the closure operator cl_{S, d_∞} agrees with the topological closure in $\pi_1(X, x_0)$.

4.2. The subgroup $D \leq \pi_1(\mathbb{D}, d_0)$ and the unique path lifting property.

Definition 4.7. Let $D \leq \pi_1(\mathbb{D}, d_0)$ be the subgroup consisting of homotopy classes of finite concatenations $\prod_{k=1}^m \ell_{n_k, j_k}^{\epsilon_k}$, $\epsilon_k \in \{\pm 1\}$ of standard paths. Recall that $d_\infty = [\lambda_1 \cdot \lambda_\infty^-]$.

Lemma 4.8. D is generated by the homotopy classes of all well-defined loops of the form $\lambda_{n,m} \cdot \lambda_{n',m'}^-$.

Proof. Let $a = \prod_{i=1}^K [\ell_{n_i, j_i}]^{\epsilon_i}$, $\epsilon_i \in \{\pm 1\}$ be a non-trivial element of D . Note that $K \geq 3$, $\epsilon_1 = 1 = -\epsilon_K$, and $j_1 = 1 = j_K$. For any dyadic unital pair (n, j) , we have $\ell_{n,j} \simeq \lambda_{n,j-1}^- \cdot \lambda_{n,j}$, $\lambda_{n,1} = \ell_{n,1}$, and $\lambda_{n,0}$ is constant. Thus a may be written as the product

$$[\ell_{n_1,1}] \left(\prod_{i=2}^{K-1} [\ell_{n_i, j_i}]^{\epsilon_i} \right) [\ell_{n_K,1}^-] = [\lambda_{n_1,1}] \left(\prod_{i=2}^{K-1} [\lambda_{n_i, j_i-1}^- \cdot \lambda_{n_i, j_i}]^{\epsilon_i} \right) [\lambda_{n_K,1}^-]$$

the latter of which is a product of elements of the form $[\lambda_{n,m} \cdot \lambda_{n',m'}^-]$. \square

Remark 4.9. Observe that $S \leq D$. Therefore a subgroup $H \leq \pi_1(X, x_0)$ is (D, d_∞) -closed whenever H is (S, d_∞) -closed.

The proof of the following proposition is nearly identical to that of Proposition 4.4.

Proposition 4.10. (D, d_∞) is a normal closure pair for (\mathbb{D}, d_0) .

Consider the subset $G = \bigcup_{n \geq 1} \mathbb{D}(n) \subseteq \mathbb{D}$ topologized as the direct limit of the inclusions

$$\mathbb{D}(1) \rightarrow \mathbb{D}(1) \cup \mathbb{D}(2) \rightarrow \mathbb{D}(1) \cup \mathbb{D}(2) \cup \mathbb{D}(3) \rightarrow \dots$$

of finite graphs. With this weak CW-topology, which is finer than the subspace topology of \mathbb{D} , G becomes a graph whose vertex set V is indexed by the set of dyadic rationals in $[0, 1]$. The unique edge between vertices $(\frac{j-1}{2^{n-1}}, 0)$ and $(\frac{j}{2^{n-1}}, 0)$ is the semicircle $\mathbb{D}(n, j)$. Note the continuous inclusion $i : G \rightarrow \mathbb{D}$ is not an embedding but does induce a monomorphism $i_\# : \pi_1(G, d_0) \rightarrow \pi_1(\mathbb{D}, d_0)$. By construction, D is precisely the image of $i_\#$. A maximal tree $T \subseteq G$ is obtained by removing all

edges $\mathbb{D}(n, k)$ where k is even (see Figure 6). According to classical graph theory, the edges of G which are not also edges of T are in bijective correspondence with generators of the free group $\pi_1(G, d_0)$. Thus generators of $\pi_1(G, d_0)$ are in bijective correspondence with the set of edges $\{\mathbb{D}(n+1, 2j) | (n, j) \in \mathcal{D}\}$. It follows that $D \cong \pi_1(G, d_0)$ is isomorphic to the free group $F(\mathcal{D})$ on the countably infinite set \mathcal{D} of dyadic unital pairs.

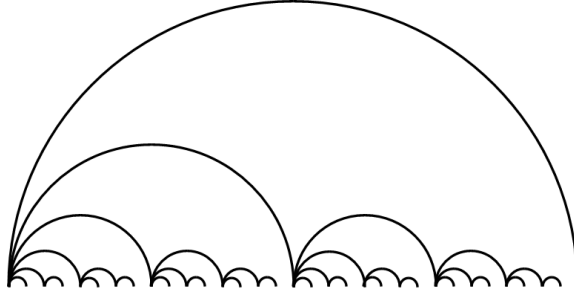


FIGURE 6. The maximal tree $T \subseteq G$

To identify explicit free generators of D , we define, for each $t \in [0, 1]$, a path $\delta_t : [0, 1] \rightarrow \mathbb{D}$ from $(0, 0)$ to $(t, 0)$. For $t = 1$, we set $\delta_1 = \ell_{1,1}$. If $t < 1$, recall that t has a binary expansion $0.a_1a_2a_3\dots = \sum_{n=1}^{\infty} \frac{a_n}{2^n}$, $a_n \in \{0, 1\}$ such that (a_n) has a cofinal subsequence of 0s. Note that t is a dyadic rational if and only if the sequence (a_n) is eventually constant at 0. We define δ_t to be the infinite concatenation $\prod_{n=1}^{\infty} \eta_n$ of a sequence of paths η_n constructed as follows: If $a_1 = 0$, let η_1 be the constant path at $(0, 0)$ and if $a_1 = 1$, let $\eta_1 = \ell_{2,1}$. Inductively, suppose the paths $\eta_1, \dots, \eta_{n-1}$ have been defined so that the concatenation $\prod_{k=1}^{n-1} \eta_k$ is a path in $T \cap \bigcup_{j=1}^n \mathbb{D}(j)$ from d_0 to $(\sum_{k=1}^{n-1} \frac{a_k}{2^k}, 0)$. Write $\sum_{k=1}^{n-1} \frac{a_k}{2^k} = \frac{j-1}{2^{n-1}}$ for $j \in \{1, \dots, 2^{n-1}\}$. If $a_n = 0$, let η_n be the constant path at $(\frac{j-1}{2^{n-1}}, 0)$ and if $a_n = 1$, set $\eta_n = \ell_{n+1, 2j-1}$. By construction, η_n has image in $T \cap \mathbb{D}(n+1)$ and $\eta_n(1) = (\sum_{k=1}^n \frac{a_k}{2^k}, 0)$. It follows that the sequence of paths η_n is null at $(t, 0)$ so that the infinite concatenation $\delta_t = \prod_{n=1}^{\infty} \eta_n$ is well-defined.

Note that δ_0 is the constant path at d_0 and if $t \in (0, 1)$ is a dyadic rational, then there is an N such that $\eta_n = c_{(t,0)}$ is the constant path at $(t, 0)$ for all $n \geq N$. In this case, δ_t is a reparameterization of the arc in T from d_0 to $(t, 0)$. We conclude that for each dyadic unital pair (n, j) , there is a corresponding free generator $d_{n,j}$ of $\pi_1(G, d_0)$ (see Figure 7) defined as the homotopy class of the loop $\left(\delta_{\frac{2j-1}{2^n}} \right) \cdot (\ell_{n+1, 2j}) \cdot \left(\delta_{\frac{j}{2^{n-1}}} \right)^{-}$.

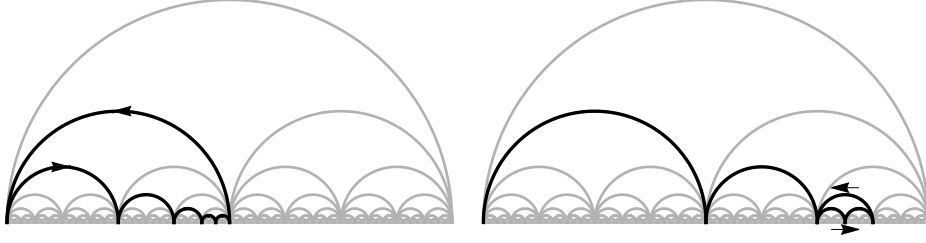


FIGURE 7. Representatives of the free generators $d_{5,8}$ (left) and $d_{4,7}$ (right).

Lemma 4.11. D is (D, d_∞) -dense in $\pi_1(\mathbb{D}, d_0)$.

Proof. To verify the sufficient condition in Remark 2.12, we show that for every loop $\alpha : ([0, 1], 0) \rightarrow (\mathbb{D}, d_0)$, there is a continuous map $f : (\mathbb{D}, d_0) \rightarrow (\mathbb{D}, d_0)$ such that $f_\#(d_\infty) = [\alpha]$ and $f_\#(D) \leq D$. To begin, we identify the domain of α with $B = [0, 1] \times \{0\} \subseteq \mathbb{D}$ and define $f(t) = \alpha(t)$ for $t \in [0, 1]$ and $f(t) = d_0$ for $t \in \mathbb{D}(1, 1)$.

For each $a, b \in [0, 1]$ with $a \leq b$ and each $n \in \mathbb{N}$ we define a path $\beta_n^{a,b}$ in $\bigcup_{k=1}^\infty \bigcup_{j=1}^{2^{k-1}} \mathbb{D}(k, j) \subseteq \mathbb{D}$ from a to b as follows: Let $i, j \in \mathbb{N}$ with $\frac{i-1}{2^{n-1}} \leq a < \frac{i}{2^{n-1}}$ and $\frac{j-1}{2^{n-1}} \leq b < \frac{j}{2^{n-1}}$. Using $T_{n,k}(x, y) = (2^{n-1}x - k + 1, 2^{n-1}y)$, we define $\beta_n^{a,b} = (T_{n,i}^{-1} \circ \delta_{T_{n,i}(a,0)})^- \cdot \ell_{n,i} \cdot \ell_{n,i+1} \cdots \ell_{n,j-1} \cdot T_{n,j}^{-1} \circ \delta_{T_{n,j}(b,0)}$. We also define $\beta_n^{b,a} = (\beta_n^{a,b})^-$.

Let \mathcal{I} be the set of components of $\alpha^{-1}(\mathbb{D} \setminus [0, 1])$. We now define f on each $\mathbb{D}(n, j)$ with $n \geq 2$ and $1 \leq j \leq 2^{n-1}$. If $\frac{j-1}{2^{n-1}} \in I$ for some interval $I \in \mathcal{I}$ with endpoints $c < d$, then put $u = \min\{d, \frac{j}{2^{n-1}}\}$; otherwise put $u = \frac{j-1}{2^{n-1}}$. Likewise, if $\frac{j}{2^{n-1}} \in J$ for some interval $J \in \mathcal{I}$ with endpoints $s < t$, then put $v = \max\{s, \frac{j-1}{2^{n-1}}\}$; otherwise put $v = \frac{j}{2^{n-1}}$. Now, let $(x, y) \in \mathbb{D}(n, j)$. We define $f(x, y) = \alpha(x)$ if $x \in [\frac{j-1}{2^{n-1}}, u] \cup [v, \frac{j}{2^{n-1}}]$. We define $f(x, y) = \beta_n^{\alpha(u), \alpha(v)}(x)$ if $x \in [u, v]$.

Clearly, $f : \mathbb{D} \rightarrow \mathbb{D}$ is well-defined. Continuity of f follows from the uniform continuity of α and the fact that $\text{diam}(\beta_n^{a,b}) \leq |a - b| + \frac{3}{2^n}$. Also, by construction, we have $f_\#(d_\infty) = [\alpha]$.

It remains to show that $f_\#(D) \leq D$. To this end, let $[p] \in D$ for some finite edge path $p = \ell_{n_1, j_1}^{\epsilon_1} \ell_{n_2, j_2}^{\epsilon_2} \cdots \ell_{n_K, j_K}^{\epsilon_K}$ in G , with $\epsilon_k \in \{+, -\}$. In order to show that $f \circ p$ is homotopic to a finite edge path in G , let u_k and v_k be defined as above for $\mathbb{D}(n_k, j_k)$. Then each $f \circ \ell_{n_k, j_k}$ is homotopic to $\alpha|_{[\frac{j_k-1}{2^{n_k-1}}, u_k]} \cdot \beta_{n_k}^{\alpha(u_k), \alpha(v_k)} \cdot \alpha|_{[v_k, \frac{j_k}{2^{n_k-1}}]}$.

Observe that if $f \circ \ell_{n_k, j_k}^{\epsilon_k}$ terminates in a nondegenerate α -segment, then $f \circ \ell_{n_{k+1}, j_{k+1}}^{\epsilon_{k+1}}$ either begins with or equals a nondegenerate α -segment. Further, every maximal concatenation of contiguous α -segments of $f \circ p$ forms a path within one and the same $\mathbb{D}(n, j)$, starting and ending in $[0, 1]$. Hence, each such subpath of $f \circ p$ can be homotoped to either $\ell_{n, j}^\pm$ or a constant. As for the β -segments, they might be “wild” at either end. However, if $u_k > \frac{j_k-1}{2^{n_k-1}}$, then the corresponding endpoint of the β -segment is a dyadic rational in $[0, 1]$, making it “tame” at that end. The same is true at the other end if $v_k < \frac{j_k}{2^{n_k-1}}$. Therefore, we only need to consider points b that are the endpoints of two consecutive β -segments such that b is a

dyadic irrational and $b = \alpha(t) \in [0, 1]$ with $t = \frac{s-1}{2^{n-1}}$ for some s . In this case, however, the two β -segments meeting in b cancel over their “wild ends”, as can be seen from the following formulas: $T_{n,j}^{-1} \circ \delta_{T_{n,j}(b,0)} = T_{n+1,2j-1}^{-1} \circ \delta_{T_{n+1,2j-1}(b,0)}$ if $\frac{j-1}{2^{n-1}} = \frac{2j-2}{2^n} < b < \frac{2j-1}{2^n}$, and $T_{n,j}^{-1} \circ \delta_{T_{n,j}(b,0)} = \ell_{n+1,2j-1} \cdot (T_{n+1,2j}^{-1} \circ \delta_{T_{n+1,2j}(b,0)})$ if $\frac{2j-1}{2^n} < b < \frac{2j}{2^n} = \frac{j}{2^{n-1}}$. \square

We use the closure pair (D, d_∞) to characterize the subgroups $H \leq \pi_1(X, x_0)$ for which $p_H : \tilde{X}_H \rightarrow X$ has the unique path lifting property. Recall the construction of \tilde{X}_H from Section 1.

Lemma 4.12. *Let $(t, 0) \in B$, $\epsilon > 0$, and $V = \mathbb{D} \cap E$ where $E \subseteq \mathbb{R}^2$ is the open disk of radius ϵ centered at $(t, 0)$. If $|s - t| < \epsilon$, then $D[\delta_s] \in B(D[\delta_t], V)$.*

Proof. Set $I = \{s \in [0, 1] \mid |s - t| < \epsilon\}$ so that $I \times \{0\} = V \cap B$. First, suppose $u, v \in I$ are dyadic rationals. In this case, δ_u and δ_v are homotopic to a finite concatenation of standard paths. Additionally, there is an arc $\gamma : [0, 1] \rightarrow V$ which is a finite concatenation of standard paths (with image in V) from $(u, 0)$ to $(v, 0)$. Since the loop $\delta_u \cdot \gamma \cdot (\delta_v)^-$ is a finite concatenation of standard paths, we have $[\delta_u \cdot \gamma \cdot \delta_v^-] \in D$. Thus $D[\delta_u] \in B(D[\delta_v], V)$. It follows that $B(D[\delta_u], V) = B(D[\delta_v], V)$ for all dyadic rationals $u, v \in I$. It now suffices to show that for each dyadic irrational $s \in I$, there is a dyadic rational $u \in I$ such that $D[\delta_s] \in B(D[\delta_u], V)$.

If $s \in I$ is not a dyadic rational, then δ_s is an infinite concatenation $\prod_{n=1}^\infty \eta_n$ from d_0 to $(s, 0) \in V$ such that η_n is either a standard path or a constant path. Find $N > 1$ such that η_n has image in V for each $n \geq N$ and let γ be the path $\prod_{n=N}^\infty \eta_n$. Note that $\gamma(0) = (u, 0)$ with $u \in I$ a dyadic rational and that δ_u is a reparameterization of $\prod_{n=1}^{N-1} \eta_n$. Since $[\delta_s \cdot \gamma^- \cdot \delta_u^-] = \left[\left(\prod_{n=1}^{N-1} \eta_n \right) \cdot \delta_u^- \right] = 1 \in D$ where γ^- has image in V , we have $D[\delta_s] \in B(D[\delta_u], V)$. \square

Theorem 4.13. *If $p_H : \tilde{X}_H \rightarrow X$ has the unique path lifting property, then $H \leq \pi_1(X, x_0)$ is (D, d_∞) -closed. The converse holds if the path space $P(X)$ is first countable, for instance, if X is metrizable.*

Proof. Suppose that H is not (D, d_∞) -closed. Then there is a map $f : (\mathbb{D}, d_0) \rightarrow (X, x_0)$ such that $f_\#(D) \leq H$ and $f_\#(d_\infty) \notin H$. We show the path $\gamma(t) = f(t, 0)$ does not have a unique lift with respect to $p_H : \tilde{X}_H \rightarrow X$. Take $\tilde{\gamma}_\mathcal{S} : [0, 1] \rightarrow \tilde{X}_H$ to be the standard lift of γ . We define a second lift $\beta : [0, 1] \rightarrow \tilde{X}_H$ by setting $\beta(t) = H[f \circ \delta_t]$. Observe that $p_H \circ \beta = \gamma$, $\beta(0) = \tilde{x}_H = \tilde{\gamma}_\mathcal{S}(0)$. Moreover, since $f_\#(d_\infty) = [f \circ (\lambda_1 \cdot \lambda_\infty^-)] \notin H$, we have $\beta(1) = H[f \circ \lambda_1] \neq H[f \circ \lambda_\infty] = H[\gamma] = \tilde{\gamma}_\mathcal{S}(1)$. Therefore, it suffices to verify the continuity of β .

Suppose $B(H[f \circ \delta_t], U)$ is an open neighborhood of $\beta(t) = H[f \circ \delta_t]$ in \tilde{X}_H where U is an open neighborhood of $f \circ \delta_t(1) = f(t, 0)$ in X . Since f is continuous, there is an $\epsilon > 0$ such that if $E \subseteq \mathbb{R}^2$ is the open disk of radius ϵ centered at $(t, 0)$, then $V = \mathbb{D} \cap E \subseteq f^{-1}(U)$. If $|s - t| < \epsilon$, then $D[\delta_s] \in B(D[\delta_t], V)$ by Lemma 4.12. Thus $[\delta_s \cdot \zeta^- \cdot \delta_t^-] \in D$ for some path ζ in V . Applying the homomorphism $f_\#$, we see that $[(f \circ \delta_s) \cdot (f \circ \zeta^-) \cdot (f \circ \delta_t)^-] = f_\#([\delta_s \cdot \zeta^- \cdot \delta_t^-]) \in f_\#(D) \leq H$ where $f \circ \zeta^-$ is a path in $f(V) \subseteq U$. It follows that $\beta(s) = H[f \circ \delta_s] \in B(H[f \circ \delta_t], U)$.

For the converse, we assume that $P(X)$ is first countable and that $p_H : \tilde{X}_H \rightarrow X$ does not have the unique path lifting property. Then there are paths $\alpha \in P(X, x_0)$ and $\beta \in P(\tilde{X}_H, \tilde{x}_H)$ such that $p_H \circ \beta = \alpha$ and $\beta(1) \neq \tilde{\alpha}_\mathcal{S}(1)$. Note that $\beta(t) =$

$H[\beta_t]$ for some path $\beta_t : [0, 1] \rightarrow X$ from x_0 to $\alpha(t)$. Thus $H[\beta_1] \neq H[\alpha]$. Since $H[\beta_0] = \tilde{x}_H$, we may assume that $\beta_0 = c_{x_0}$. We extend α to a map $f : \mathbb{D} \rightarrow X$ such that $f((u, 0)) = \alpha(u)$ on the base arc. Consider a countable neighborhood base $\mathcal{U}_1 \supset \mathcal{U}_2 \supset \dots$ at α in $P(X)$. We may assume that \mathcal{U}_p is of the form

$$\mathcal{U}_p = \bigcap_{j=1}^{2^{n(p)-1}} \left\langle \left[\frac{j-1}{2^{n(p)-1}}, \frac{j}{2^{n(p)-1}} \right], U_{n(p),j} \right\rangle$$

where $1 = n(1) < n(2) < \dots$ is an increasing sequence of natural numbers.

To define f on the $n(p)$ -th level, we take the approach of [22, Theorem 2.9]. Suppose $n \in \mathbb{N}$ is such that $n = n(p)$ for some p . Since $\beta_t(1) = \alpha(t)$ for each $t \in [0, 1]$, we have $\beta(t) = H[\beta_t] \in B(H[\beta_t], U_{n,j})$ for each $t \in [\frac{j-1}{2^{n-1}}, \frac{j}{2^{n-1}}]$. Therefore, there is a subdivision $\frac{j-1}{2^{n-1}} = s_0 < s_1 < \dots < s_k = \frac{j}{2^{n-1}}$ such that $\beta([s_{i-1}, s_i]) \subseteq B(H[\beta_{s_{i-1}}], U_{n,j})$ for each $i = 1, \dots, k$. In particular, there is a path $\zeta_i : [0, 1] \rightarrow U_{n,j}$ from $\alpha(s_{i-1})$ to $\alpha(s_i)$ such that $H[\beta_{s_{i-1}} \cdot \zeta_i] = H[\beta_{s_i}]$.

Note that the concatenation $\alpha_{n,j} = \zeta_1 \cdot \zeta_2 \cdot \dots \cdot \zeta_k$ is a path in $U_{n,j}$ from $\alpha(\frac{j-1}{2^{n-1}})$ to $\alpha(\frac{j}{2^{n-1}})$. Since $[\beta_{s_{i-1}} \cdot \zeta_i \cdot \beta_{s_i}^-] \in H$, we have

$$\left[\beta_{\frac{j-1}{2^{n-1}}} \cdot \alpha_{n,j} \cdot \beta_{\frac{j}{2^{n-1}}}^- \right] = \prod_{i=1}^k [\beta_{s_{i-1}} \cdot \zeta_i \cdot \beta_{s_i}^-] \in H$$

for each $j = 1, \dots, 2^{n-1}$. Set $\alpha_n = \prod_{j=1}^{2^{n-1}} \alpha_{n,j}$.

If $n \in \mathbb{N}$ is such that $n(p) < n < n(p+1)$, put $\alpha_n = \alpha_{n(p+1)}$. By construction, the sequence α_n converges to α and satisfies $\alpha_n(\frac{j}{2^{n-1}}) = \alpha(\frac{j}{2^{n-1}})$ for all $n \geq 1$, $0 \leq j \leq 2^{n-1}$. Thus we obtain a unique map $f : \mathbb{D} \rightarrow X$ such that $f \circ \lambda_n = \alpha_n$ and $f \circ \lambda_\infty = \alpha$. Observe that $f \circ \ell_{n,j} = \alpha_{n,j}$ whenever $n = n(p)$ for some $p \in \mathbb{N}$ and $1 \leq j \leq 2^{n-1}$.

Finally, we check that $f_\#(D) \leq H$ and $f_\#(d_\infty) \notin H$. First, we claim that $H[f \circ \lambda_{n,m}] = H[\beta_{\frac{m}{2^{n-1}}}]$ for each dyadic unital pair (n, m) . Since

$$f \circ \lambda_{n,m} = f \circ \left(\prod_{j=1}^m \ell_{n,j} \right) = f \circ \left(\prod_{j=1}^{m2^{n(p)-n}} \ell_{n(p),j} \right) = f \circ \lambda_{n(p), m2^{n(p)-n}}$$

for some $n(p) \geq n$ and $p \geq 1$, we may assume that $n = n(p)$. In this case, $f \circ \lambda_{n,m} = \prod_{j=1}^m \alpha_{n,j}$. We have

$$\begin{aligned} \left[(f \circ \lambda_{n,m}) \cdot \left(\beta_{\frac{m}{2^{n-1}}} \right)^- \right] &= \left[\beta_0 \cdot (f \circ \lambda_{n,m}) \cdot \left(\beta_{\frac{m}{2^{n-1}}} \right)^- \right] \\ &= [\beta_0] \left(\prod_{j=1}^m [\alpha_{n,j}] \right) \left[\left(\beta_{\frac{m}{2^{n-1}}} \right)^- \right] \\ &= \prod_{j=1}^m \left[\beta_{\frac{j-1}{2^{n-1}}} \cdot \alpha_{n,j} \cdot \left(\beta_{\frac{j}{2^{n-1}}} \right)^- \right] \in H \end{aligned}$$

showing that $H[f \circ \lambda_{n,m}] = H[\beta_{\frac{m}{2^{n-1}}}]$ as desired.

Whenever $\lambda_{m,n} \cdot \lambda_{m',n'}^-$ is a well-defined loop, we have $\frac{m}{2^n-1} = \frac{m'}{2^{n'}-1}$ and thus

$$H[f \circ \lambda_{n,m}] = H\left[\beta_{\frac{m}{2^n-1}}\right] = H\left[\beta_{\frac{m'}{2^{n'}-1}}\right] = H[f \circ \lambda_{n',m'}]$$

This proves $f_{\#}([\lambda_{m,n} \cdot \lambda_{m',n'}^-]) \in H$ whenever the loop is defined. Since the elements $[\lambda_{m,n} \cdot \lambda_{m',n'}^-]$ generate D by Lemma 4.8, we have $f_{\#}(D) \leq H$. Finally, since $H[f \circ \lambda_1] = H[\beta_1] \neq H[\alpha]$, we have $f_{\#}(d_{\infty}) = f_{\#}([\lambda_1 \cdot \lambda_{\infty}^-]) = [(f \circ \lambda_1) \cdot \alpha^-] \notin H$. \square

If X is a one-dimensional metric space, the following Theorem of Eda implies that every homomorphism $\pi_1(\mathbb{D}, d_0) \rightarrow \pi_1(X, x_0)$ is induced by a continuous map up to a change of basepoint. In this case, the unique path lifting property for $H \leq \pi_1(X, x_0)$ depends only on group theoretic conditions. If $\gamma : [0, 1] \rightarrow X$ is a path from x_0 to x_1 , let $\phi_{\gamma} : \pi_1(X, x_0) \rightarrow \pi_1(X, x_1)$ be the conjugation isomorphism $\phi_{\gamma}([\alpha]) = [\gamma^- \cdot \alpha \cdot \gamma]$.

Theorem 4.14. [17] *Let Y be a one-dimensional Peano continuum, X be a one-dimensional metric space, $x_0 \in X$, and $y_0 \in Y$. For each homomorphism $h : \pi_1(Y, y_0) \rightarrow \pi_1(X, x_0)$, there exists a continuous map $f : Y \rightarrow X$ and a path $\gamma : [0, 1] \rightarrow X$ from x_0 to $f(y_0)$ such that $\phi_{\gamma} \circ h = f_{\#}$.*

Corollary 4.15. *Suppose X is a one-dimensional metric space and $H \leq \pi_1(X, x_0)$ is a subgroup. Then $p_H : \tilde{X}_H \rightarrow X$ has the unique path lifting property if and only if every homomorphism $h : \pi_1(\mathbb{D}, d_0) \rightarrow \pi_1(X, x_0)$ satisfying $h(D) \leq H$ has image in H .*

Proof. One direction follows immediately from Theorem 4.13. If $p_H : \tilde{X}_H \rightarrow X$ has the unique lifting property, then H is (D, d_{∞}) -closed. Let $h : \pi_1(\mathbb{D}, d_0) \rightarrow \pi_1(X, x_0)$ be any homomorphism satisfying $h(D) \leq H$. Since \mathbb{D} is a one-dimensional Peano continuum, by Theorem 4.14, there is a map $f : \mathbb{D} \rightarrow X$ and a path $\gamma : [0, 1] \rightarrow X$ from x_0 to $f(d_0)$ such that $\phi_{\gamma} \circ h = f_{\#}$. By Proposition 4.10, the conjugate subgroup $H^{\gamma} = [\gamma^-]H[\gamma]$ is (D, d_{∞}) -closed. Since $f_{\#}(D) = \phi_{\gamma}(h(D)) \leq \phi_{\gamma}(H) = H^{\gamma}$ and D is (D, d_{∞}) -dense in $\pi_1(\mathbb{D}, d_0)$, we have $f_{\#}(\pi_1(\mathbb{D}, d_0)) \leq H^{\gamma}$. It follows that $h(\pi_1(\mathbb{D}, d_0)) = \phi_{\gamma}^{-1}(f_{\#}(\pi_1(\mathbb{D}, d_0))) \leq \phi_{\gamma}^{-1}(H^{\gamma}) = H$. \square

5. INTERMEDIATE GENERALIZED COVERINGS

We begin our discussion of intermediate generalized coverings by contrasting it with the situation for traditional covering maps. Call a subgroup $H \leq \pi_1(X, x_0)$ a (generalized) covering subgroup if there is a (generalized) covering map $p : (\hat{X}, \hat{x}) \rightarrow (X, x_0)$ such that $p_{\#}(\pi_1(\hat{X}, \hat{x})) = H$. Recalling Theorem 1.3, it is a classical result of covering space theory that if X is locally path connected then a subgroup $H \leq \pi_1(X, x_0)$ is a covering subgroup if and only if for every point $x \in X$, there is an open neighborhood $U_x \in \mathcal{T}_x$ such that the normal subgroup $N = \langle \pi(x, U_x) | x \in X \rangle \trianglelefteq \pi_1(X, x_0)$ is contained in H . In particular, N itself is a covering subgroup and so is every subgroup K with $N \leq K \leq \pi_1(X, x_0)$. Therefore the collection of covering subgroups of $\pi_1(X, x_0)$ is upwards closed in the subgroup lattice of $\pi_1(X, x_0)$.

It also follows from the previous paragraph that the collection of covering subgroups is closed under finite intersection and that the core of a covering subgroup H

is a covering subgroup of $\pi_1(X, x_0)$. Despite these special cases, covering subgroups are not closed under arbitrary intersection. For example,

$$\bigcap \{N \trianglelefteq \pi_1(\mathbb{H}, b_0) \mid N \text{ is a covering subgroup}\} = 1$$

but \mathbb{H} does not admit a universal covering space.

The situation for generalized coverings is quite different. The collection of generalized covering subgroups is closed under arbitrary intersection [3] but is not upwards closed in the subgroup lattice of $\pi_1(X, x_0)$. For example, $1 \leq \pi_1(\mathbb{H}, b_0)$ is a generalized covering subgroup since \mathbb{H} admits a generalized universal covering, while the free subgroup $F = \langle [\ell_n] \mid n \in \mathbb{N} \rangle \leq \pi_1(\mathbb{H}, b_0)$ is not a generalized covering subgroup [24]. On the other hand, the core of a generalized covering subgroup is always a generalized covering subgroup, because it equals the intersection of conjugate generalized covering subgroups. In Theorem 5.4 below, we identify a condition sufficient to conclude that if N is a normal, generalized covering subgroup and $N \leq H$, then H is also a generalized covering subgroup.

Example 5.1. Recall the (C, c_∞) -closed subgroup $CCP(\mathbb{D}, B, d_0) \leq \pi_1(\mathbb{D}, d_0)$ constructed in Example 3.11. We claim that $CCP(\mathbb{D}, B, d_0)$ is not (D, d_∞) -closed. Since $D \leq CCP(\mathbb{D}, B, d_0)$, it suffices to check that $d_\infty \notin CCP(\mathbb{D}, B, d_0)$. If $d_\infty = [\lambda_1 \cdot \lambda_\infty^-] \in CCP(\mathbb{D}, B, d_0)$, then there is a loop β which is homotopic to the reduced loop $\alpha = \lambda_1 \cdot \lambda_\infty^-$ such that $\beta^{-1}(B)$ is countable. However the reduced loop α must be obtained by contracting subpaths of β within its own image. Thus $\alpha^{-1}(B)$ must be countable; a contradiction.

Example 5.2. The construction of a normal subgroup $N \trianglelefteq \pi_1(\mathbb{D}, d_0)$ with the same properties as the subgroup in the previous example is a bit more involved. However this is done in [27] using a triangle-space T , which is homotopy equivalent to \mathbb{D} . We refer to this paper for detailed proofs. Call a path $\alpha : [0, 1] \rightarrow \mathbb{D}$ *generic* if there exists a countable, closed set $A \subset [0, 1]$ such that the set of components of $[0, 1] \setminus A$ may be written as a disjoint union $C_0 \cup C_1^+ \cup C_1^-$ where

- (1) if $(a, b) \in C_0$, then $\alpha([a, b]) \cap B$ is closed and nowhere dense in B ,
- (2) there is bijection $\theta : C_1^+ \rightarrow C_1^-$ such that if $\theta((a, b)) = (c, d)$, then $\alpha|_{[a, b]} \equiv (\alpha|_{[c, d]})^-$.

It is straightforward to show that $N = \{[\alpha] \in \pi_1(\mathbb{D}, d_0) \mid \alpha \text{ is generic}\}$ is a normal, (C, c_∞) -closed subgroup of $\pi_1(\mathbb{D}, d_0)$ and thus (C, c_τ) -closed by Proposition 3.8. While $D \leq N$, the arguments in [27] show that $d_\infty \notin N$. Therefore, N is not (D, d_∞) -closed.

If $H, K \leq G$ are subgroups, let $K^H = \bigcup_{h \in H} h^{-1}Kh$. For instance, recall that $\pi(x, U) = \langle \pi(\alpha, U)^{\pi_1(X, x_0)} \rangle = \langle \pi(\beta, U) \mid \beta(1) = x \rangle$ is the normal closure of $\pi(\alpha, U)$ in $\pi_1(X, x_0)$.

Proposition 5.3. *Suppose X is a metric space and $N \leq H \leq \pi_1(X, x_0)$ where N is a normal subgroup of $\pi_1(X, x_0)$ and $p_N : \tilde{X}_N \rightarrow X$ has the unique path lifting property. If, for every path $\alpha : [0, 1] \rightarrow X$ with $\alpha(0) = x_0$, there is a $U \in \mathcal{T}_{\alpha(1)}$ such that $\pi(\alpha, U)^H \cap H \subseteq N$, then $p_H : \tilde{X}_H \rightarrow X$ has the unique path lifting property.*

Proof. Let $f : (\mathbb{D}, d_0) \rightarrow (X, x_0)$ be such that $f_\#(D) \leq H$. By Theorem 4.13, it suffices to show that there is an $m \geq 1$ such that $f_\#([\lambda_m \cdot \lambda_\infty^-]) \in H$. Identify $[0, 1]$ with the base arc B . For each $(n, j) \in \mathcal{D}$, let $\mathbb{D}_{n,j}$ be the homeomorphic copy of \mathbb{D}

bounded by the arc $\mathbb{D}(n, j)$ and the interval $[\frac{j-1}{2^{n-1}}, \frac{j}{2^{n-1}}] \subseteq B$. Recall the canonical homeomorphism $T_{n,j} : \mathbb{D}_{n,j} \rightarrow \mathbb{D}$ from Lemma 4.11.

For each $t \in B$ there is an open neighborhood U_t of $f(t)$ such that $\pi(f \circ \delta_t, U_t)^H \cap H \subseteq N$. Let V_t be an open ball centered at t in \mathbb{D} such that $f(V_t) \subseteq U_t$. Take $0 \leq t_1 < t_2 < \dots < t_k \leq 1$ such that the sets $V_{t_1}, V_{t_2}, \dots, V_{t_k}$ cover B . There exists an $m \geq 1$ such that for each $j = 1, \dots, 2^{m-1}$, we have $\mathbb{D}_{m,j} \subseteq V_{t_i}$ for some i . For the moment, fix such j and i . Put $s = \frac{j-1}{2^{m-1}}$, $\alpha_j = f \circ \delta_s$, and let $f_j : \mathbb{D}_{m,j} \rightarrow X$ denote the restriction of f . By Lemma 4.12, there is a path $\epsilon : [0, 1] \rightarrow V_{t_i}$ from t_i to s and an element $L \in D$ such that $[\delta_s] = L^{-1}[\delta_{t_i} \cdot \epsilon]$. Consider a free generator $d_{n,k} = [\delta_{q_1} \cdot \ell_{n+1,2k} \cdot \delta_{q_2}^-]$ of D where $q_1 = \frac{2k-1}{2^n}$ and $q_2 = \frac{k}{2^{n-1}}$. We have $[\delta_s](T_{m,j}^{-1})_{\#}(d_{n,k})[\delta_s^-] \in D$ since $\delta_s \cdot T_{m,j}^{-1} \circ (\delta_{q_1} \cdot \ell_{n+1,2k} \cdot \delta_{q_2}^-) \cdot \delta_s^-$ is a finite concatenation of standard paths. Also,

$$[\delta_s](T_{m,j}^{-1})_{\#}(d_{n,k})[\delta_s^-] = L^{-1}[\delta_{t_i}][\epsilon](T_{m,j}^{-1})_{\#}(d_{n,k})[\epsilon^-][\delta_{t_i}^-]L$$

is an element of $L^{-1}\pi(\delta_{t_i}, V_{t_i})L \subseteq \pi(\delta_{t_i}, V_{t_i})^D$. Applying the homomorphism $f_{\#}$ and recalling that $f_{\#}(D) \leq H$, we have

$$[\alpha_j](f_j \circ T_{m,j}^{-1})_{\#}(d_{n,k})[\alpha_j^-] \in \pi(f \circ \delta_{t_i}, U_{t_i})^H \cap H \subseteq N.$$

Thus $f_j \circ T_{m,j}^{-1} : \mathbb{D} \rightarrow X$ is a map satisfying $(f_j \circ T_{m,j}^{-1})_{\#}(D) \leq N^{\alpha_j}$ for $1 \leq j \leq 2^{m-1}$. Since N is (D, d_{∞}) -closed by assumption, N^{α_j} is (D, d_{∞}) -closed by Proposition 4.10. Thus, for each j , we have $(f_j \circ T_{m,j}^{-1})_{\#}(d_{\infty}) \in N^{\alpha_j}$. Let β_j be the restriction of λ_{∞} to $[\frac{j-1}{2^{m-1}}, \frac{j}{2^{m-1}}] \subseteq B$. Then $f_{\#}([\delta_s][\ell_{m,j} \cdot \beta_j^-][\delta_s^-]) \in N$ for each $s = \frac{j-1}{2^{m-1}}$. Since N is a normal subgroup and $[\lambda_m \cdot \lambda_{\infty}^-]$ factors as a product of conjugates of the elements $[\delta_s][\ell_{m,j} \cdot \beta_j^-][\delta_s^-]$, we see that $f_{\#}([\lambda_m \cdot \lambda_{\infty}^-]) \in N \leq H$. \square

Since $\pi(\alpha, U)^H \subseteq \pi(x, U)$ for any path α with $\alpha(1) = x$ and any subgroup $H \leq \pi_1(X, x_0)$, we obtain the following theorem.

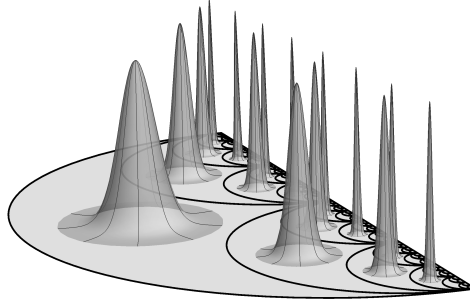
Theorem 5.4. *Suppose X is a metric space and $N \leq H \leq \pi_1(X, x_0)$ where N is a normal subgroup of $\pi_1(X, x_0)$ and $p_N : \tilde{X}_N \rightarrow X$ has the unique path lifting property. If, for every point $x \in X$, there is a $U \in \mathcal{T}_x$ such that $\pi(x, U) \cap H \leq N$, then $p_H : \tilde{X}_H \rightarrow X$ has the unique path lifting property.*

Remark 5.5. The condition given in Theorem 5.4 may be interpreted as follows: if a normal (D, d_{∞}) -closed subgroup $N \leq \pi_1(X, x_0)$ is enlarged to a subgroup $H \leq \pi_1(X, x_0)$ by adding only homotopy classes of loops which are “large” in the sense that they do not factor as products of conjugates of arbitrarily small loops based at some fixed point, then H must also be (D, d_{∞}) -closed.

Example 5.6. Since \mathbb{D} is a 1-dimensional Peano continuum it admits a generalized universal covering [24], which makes the trivial subgroup $1 \leq \pi_1(\mathbb{D}, d_0)$ a (D, d_{∞}) -closed subgroup. The subgroup $S \leq \pi_1(\mathbb{D}, d_0)$ is not (S, d_{∞}) -closed since it does not contain d_{∞} . However, $1 \leq S$ and no non-trivial element of S has a representative of the form $\prod_{i=1}^n \alpha_i \cdot \delta_i \cdot \alpha_i^-$ with loops δ_i based at the same $x \in X$ and $\text{diam}(\delta_i) < 1$. Thus Theorem 5.4 implies that S is (D, d_{∞}) -closed. We conclude that $p_S : \tilde{\mathbb{D}}_S \rightarrow \mathbb{D}$ is a generalized covering map, which is not a covering map since S does not satisfy the necessary condition given in Theorem 1.3.

6. GENERALIZED UNIVERSAL COVERINGS

Consider \mathbb{D} as the subspace $\mathbb{D} \times \{0\} \subseteq \mathbb{R}^3$. The dyadic unital pairs (n, j) are in bijective correspondence with the open disks $e_{n,j} \subseteq \mathbb{R}^2 \times \{0\}$ which are the bounded components of $(\mathbb{R}^2 \setminus \mathbb{D}) \times \{0\}$ in $\mathbb{R}^2 \times \{0\}$. We construct the space $\mathbb{D}\mathbb{A} \subseteq \mathbb{R}^3$ by continuously raising a point in each disk $e_{n,j}$ up to unit height while leaving \mathbb{D} unchanged (see Figure 8). Note that $\mathbb{D}\mathbb{A}$ is homotopy equivalent to the space obtained by attaching a 2-cell to \mathbb{D} using the loop $\ell_{n,j} \cdot (\ell_{n+1,2j})^- \cdot (\ell_{n+1,2j-1})^-$ as an attaching map for each dyadic unital pair (n, j) . Thus $\pi_1(\mathbb{D}\mathbb{A}, d_0) \cong \pi_1(\mathbb{D}, d_0)/N$ where N is the normal closure of D in $\pi_1(\mathbb{D}, d_0)$.

FIGURE 8. The space $\mathbb{D}\mathbb{A}$

Theorem 6.1. *The following are equivalent for any path-connected metric space X .*

- (1) X admits a generalized universal covering,
- (2) every map $f : \mathbb{D} \rightarrow X$ such that $f_{\#}(D) = 1$ induces the trivial homomorphism on π_1 ,
- (3) every map $g : \mathbb{D}\mathbb{A} \rightarrow X$ induces the trivial homomorphism on π_1 .

Proof. (1) \Leftrightarrow (2) is a direct combination of Theorem 4.13, Lemma 4.11, and Corollary 2.13. (2) \Leftrightarrow (3) is evident from the fact that a map $f : \mathbb{D} \rightarrow X$ extends to a map $g : \mathbb{D}\mathbb{A} \rightarrow X$ if and only if $f_{\#}(D) = 1$. \square

Definition 6.2. Let $Cov(X)$ denote the set of all open covers of X , inversely directed by refinement. For $\mathcal{U} \in Cov(X)$, the *Spanier group of X relative to \mathcal{U}* is the subgroup $\pi(\mathcal{U}, x_0) = \langle \pi(x, U) \mid x \in U \in \mathcal{U} \rangle$. The *Spanier group of X* is the intersection $\pi^s(X, x_0) = \bigcap_{\mathcal{U} \in Cov(X)} \pi(\mathcal{U}, x_0)$.

If X is locally path connected, then X is semilocally simply connected if and only if there is an open cover \mathcal{U} such that $\pi(\mathcal{U}, x_0) = 1$. It is shown in [24] that if $\pi^s(X, x_0) = 1$, then X admits a generalized universal covering. Indeed, for every path-connected space X and every open cover \mathcal{U} of X , $\pi(\mathcal{U}, x_0)$ is an (S, d_{∞}) -closed subgroup of $\pi_1(X, x_0)$ (the straightforward proof is similar to that of Proposition 6.4 below). Therefore, the intersection $\pi^s(X, x_0)$ is (S, d_{∞}) -closed. We also note that for locally path-connected X , $\pi^s(X, x_0)$ equals the kernel of the canonical homomorphism $\pi(X, x_0) \rightarrow \tilde{\pi}_1(X, x_0)$ [5].

Definition 6.3. Let G be a group.

- (1) We call G *noncommutatively slender* (or *n-slender* for short) if for every homomorphism $h : \pi_1(\mathbb{H}, b_0) \rightarrow G$, there is an N such that $h([\alpha]) = 1$ for all $\alpha \in \Omega(\mathbb{H}_{\geq N}, b_0)$; equivalently, G is n-slender if and only if for every Peano continuum X and homomorphism $h : \pi_1(X, x_0) \rightarrow G$, there exists an open cover \mathcal{U} of X such that $h(\pi(\mathcal{U}, x_0)) = 1$ [16].
- (2) We call G *residually n-slender* if for every $g \in G \setminus \{1\}$, there is an n-slender group K and a homomorphism $k : G \rightarrow K$ such that $k(g) \neq 1$.
- (3) We call G *homomorphically Hausdorff relative to a space X* if for every homomorphism $h : \pi_1(X, x_0) \rightarrow G$, we have $\bigcap_{\mathcal{U} \in \text{Cov}(X)} h(\pi(\mathcal{U}, x_0)) = 1$.

Observe that if X is path connected, locally path connected and first countable, and if $\pi_1(X, x_0)$ is n-slender, then X is semilocally simply connected so that X admits a classical universal covering.

Every free group is n-slender [13] and certainly every n-slender group is residually n-slender. If G is residually n-slender, then G is homomorphically Hausdorff relative to every Peano continuum [19]. Therefore, if X is a Peano continuum and $\pi_1(X, x_0)$ is residually n-slender, then we have $\pi^s(X, x_0) = \bigcap_{\mathcal{U} \in \text{Cov}(X)} \text{id}_{\#}(\pi(\mathcal{U}, x_0)) = 1$, which implies that X admits a generalized universal covering. Using the test space \mathbb{D} , we extend this result to all metric spaces.

Proposition 6.4. *If X is metrizable and $\pi_1(X, x_0)$ is homomorphically Hausdorff relative to \mathbb{D} , then X is homotopically path-Hausdorff.*

Proof. Suppose $\pi_1(X, x_0)$ is homomorphically Hausdorff relative to \mathbb{D} . By Theorem 6.1, we may check that the trivial subgroup is (S, d_∞) -closed. Let $f : (\mathbb{D}, d_0) \rightarrow (X, x_0)$ be a based map such that $f_{\#}(S) = 1$. Fix an open cover $\mathcal{U} \in \text{Cov}(\mathbb{D})$. There is an $n \geq 1$ such that $[\lambda_n \cdot \lambda_\infty^-] \in \pi(\mathcal{U}, d_0)$. Since $[\lambda_1 \cdot \lambda_n^-] \in S$, we have $f_{\#}(d_\infty) = f_{\#}([\lambda_1 \cdot \lambda_n^-])f_{\#}([\lambda_n \cdot \lambda_\infty^-]) = f_{\#}([\lambda_n \cdot \lambda_\infty^-]) \in f_{\#}(\pi(\mathcal{U}, d_0))$. Thus $f_{\#}(d_\infty) \in f_{\#}(\pi(\mathcal{U}, d_0))$ for every $\mathcal{U} \in \text{Cov}(\mathbb{D})$. By assumption, $\bigcap_{\mathcal{U} \in \text{Cov}(\mathbb{D})} f_{\#}(\pi(\mathcal{U}, x_0)) = 1$; therefore $f_{\#}(d_\infty) = 1$. \square

Corollary 6.5. *If X is a metric space and $\pi_1(X, x_0)$ is residually n-slender, then X admits a generalized universal covering.*

Definition 6.6. A space X is $1\text{-}UV_0$ at $x \in X$ if for every neighborhood U of x there is open set V in X with $x \in V \subseteq U$ and such that for every map $f : D^2 \rightarrow X$ with $f(S^1) \subseteq V$, there is a map $g : D^2 \rightarrow U$ with $f|_{S^1} = g|_{S^1}$. We say that X is $1\text{-}UV_0$ if X is $1\text{-}UV_0$ at every point $x \in X$.

Remark 6.7. Unlike many of the properties considered in this paper, the $1\text{-}UV_0$ property is not an invariant of homotopy type. Indeed, the cone $C\mathbb{H} = \frac{\mathbb{H} \times [0, 1]}{\mathbb{H} \times \{1\}}$ over the Hawaiian earring is homotopy equivalent to the one-point space but is not $1\text{-}UV_0$.

The authors of [10] show that X is homotopically Hausdorff at $x \in X$ whenever X is $1\text{-}UV_0$ at x . We improve upon this result in Theorem 6.9 below. Note the resemblance of the following characterization to Corollary 3.9.

Proposition 6.8. *Identify \mathbb{H} with a subspace of the unit disk D^2 so the outermost circle $C_1 \subseteq \mathbb{H}$ is identified with S^1 . For a first countable space X , the following are equivalent:*

- (1) X is $1\text{-}UV_0$ at $x \in X$,

- (2) every map $f : (\mathbb{H}, b_0) \rightarrow (X, x)$ such that $f_{\#}(F) = 1$ extends to a map $g : D^2 \rightarrow X$,
- (3) for every map $f : (\mathbb{H}\mathbb{A}, b_0) \rightarrow (X, x)$, $f|_{\mathbb{H}} : \mathbb{H} \rightarrow X$ extends to a map $g : D^2 \rightarrow X$.

Proof. The proof of (1) \Rightarrow (3) is identical to the proof of Lemma 4.1 in [10]. (3) \Rightarrow (2) is obvious since every map $f : \mathbb{H} \rightarrow X$ satisfying $f_{\#}(F) = 1$ extends to a map on the harmonic archipelago. (2) \Rightarrow (1) Suppose X is not $1-UV_0$ at x . Then there is an open neighborhood U of x , a countable basis $\dots \subseteq U_3 \subseteq U_2 \subseteq U_1 = U$ at x , and loops $\gamma_n \in \Omega(U_n, x)$ which are inessential in X but which are essential in U_1 . Define a map $f : (\mathbb{H}, b_0) \rightarrow (X, x)$ by $f \circ \ell_n = \gamma_n$ and note that $f_{\#}(F) = 1 \leq \pi_1(X, x)$. However, if f extended to a map $g : D^2 \rightarrow X$, there would be an m such that γ_m is inessential in U_1 . Thus no such g can exist. \square

Theorem 6.9. *If X is metrizable and $1-UV_0$, then X admits a generalized universal covering.*

Proof. Fix a metric generating the topology of X . By Theorem 4.13, it suffices to show that the trivial subgroup of $\pi_1(X, x_0)$ is (D, d_{∞}) -closed. Suppose X is $1-UV_0$ and $f : (\mathbb{D}, d_0) \rightarrow (X, x)$ is a map such that $f_{\#}(D) = 1$. If $\beta_{n,j} : S^1 \rightarrow \mathbb{D}$ is the loop defined as $\ell_{n,j} \cdot \ell_{n+1,2j}^- \cdot \ell_{n+1,2j-1}^-$, then $f \circ \beta_{n,j}$ is inessential in X . Let $E_{n,j}$ be the set of extensions $h : D^2 \rightarrow X$ of $f \circ \beta_{n,j}$. Suppose there exists $h_{n,j} \in E_{n,j}$ such that $s_n = \max\{\text{diam}(h_{n,j}(D^2)) \mid j = 1, \dots, 2^{n-1}\} \rightarrow 0$. In this case, f extends to a map on $\{(x, y) \in \mathbb{R}^2 \mid (x - 1/2)^2 + y^2 \leq \frac{1}{4}, y \geq 0\}$ showing that f is null-homotopic; consequently $f_{\#}(d_{\infty}) = 1$. We show the other case cannot occur. Suppose there exists $\epsilon > 0$ and $(n_k, j_k) \in \mathcal{D}$ where $n_1 < n_2 < n_3 < \dots$ such that every extension of $f \circ \beta_{n_k, j_k}$ to D^2 has diameter $> \epsilon$. Replacing (n_k, j_k) with a subsequence if necessary, we may assume that β_{n_k, j_k} converges uniformly to the constant loop at some point $(t, 0) \in B$. Pick arcs α_k from $(t, 0)$ to $\left(\frac{j_k - 1}{2^{n_k - 1}}, 0\right)$ in B and observe that $\text{diam}(\alpha_k([0, 1])) \rightarrow 0$. Define a map $f' : \mathbb{H} \rightarrow X$ by $f' \circ \ell_k = \alpha_k \cdot (f \circ \beta_{n_k, j_k}) \cdot \alpha_k^-$. It is clear that $(f')_{\#}(F) = 1$ yet f' cannot extend to a map on D^2 as in Proposition 6.8; a contradiction. \square

Example 6.10. The Peano continua A and B in [10] (also appearing in [22]) are sometimes referred to as the “sombbrero spaces.” In [10], both spaces are shown to either have the $1-UV_0$ property or a stronger property. Theorem 6.9 implies that both spaces admit a generalized universal covering space. Thus B is an example of a Peano continuum which is not homotopically path Hausdorff [22, Prop. 3.4] but which admits a generalized universal covering space.

7. TRANSFINITE PATH PRODUCTS

Definition 7.1. A space X has *transfinite path products relative to $H \leq \pi_1(X, x_0)$* if for every closed set $A \subseteq [0, 1]$ containing $\{0, 1\}$ and paths $p, q : ([0, 1], 0) \rightarrow (X, x_0)$ such that $p|_A = q|_A$ and $[p|_{[0,b]} \cdot q|_{[a,b]}^- \cdot p|_{[0,a]}^-] \in H$ for every component (a, b) of $[0, 1] \setminus A$, we have $[p \cdot q^-] \in H$. A space X has *transfinite path products* if X has transfinite path products relative to the trivial subgroup $H = 1$.

Remark 7.2. Suppose that $A \subseteq [0, 1]$ is closed and paths p, q satisfy the hypotheses of Definition 7.1. If B is the boundary of A in $[0, 1]$, then B is nowhere dense and

closed in $[0, 1]$, $p|_B = q|_B$, and if (c, d) is any component of $[0, 1] \setminus B$, then we still have $[p|_{[0, c]} \cdot q|_{[c, d]}^- \cdot p|_{[0, c]}^-] \in H$. Therefore, to verify that a space X has transfinite path products relative to H , it suffices to check the statement of the definition for nowhere dense, closed subsets $A \subseteq [0, 1]$.

Let $\mathcal{C} \subseteq [0, 1]$ be the standard middle third cantor set and \mathcal{I} be the set of components of $[0, 1] \setminus \mathcal{C}$ with the natural ordering inherited from $[0, 1]$ (which is order isomorphic to \mathbb{Q}). For each interval $I = (\frac{j-1}{3^{n-1}}, \frac{j}{3^{n-1}}) \in \mathcal{I}$, let \mathbb{W}_I be the upper-half of the circle of radius $\frac{1}{2(3^{n-1})}$ centered at $(\frac{2j-1}{2(3^{n-1})}, 0)$. Let \mathbb{W} be the union of these semicircles and the base arc $B = [0, 1] \times \{0\}$ (see Figure 9).

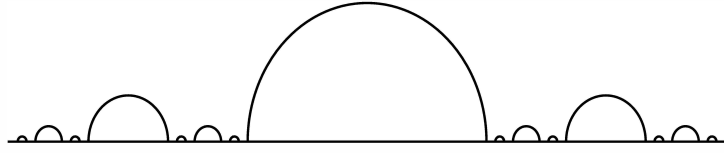


FIGURE 9. The space \mathbb{W}

Let $\lambda_\infty : [0, 1] \rightarrow \mathbb{W}$, $\lambda_\infty(t) = (t, 0)$ be the arc along the base arc and $v_\infty : [0, 1] \rightarrow \mathbb{W}$ be the arc along the upper portion of \mathbb{W} that is $v_\infty(t) = (t, 0)$ for $t \in \mathcal{C}$ and if $I = (a, b) \in \mathcal{I}$, then $v_\infty|_{[a, b]}$ is the arc along \mathbb{W}_I from $(a, 0)$ to $(b, 0)$.

Definition 7.3. Let $W \leq \pi_1(\mathbb{W}, d_0)$ be the subgroup generated by the set $\{[v_\infty|_{[0, b]} \cdot \lambda_\infty|_{[a, b]}^- \cdot v_\infty|_{[a, b]}](a, b) \in \mathcal{I}\}$ and $w_\infty = [v_\infty \cdot \lambda_\infty^-]$.

We may identify \mathbb{W}^+ as a subspace of \mathbb{W} and define a retraction $r : \mathbb{W} \rightarrow \mathbb{W}^+$ such that $r_\#(W) \leq W^+$ and $r_\#(w_\infty) = w_\infty^+$. Thus the proof of Proposition 4.4 may be modified for \mathbb{W} .

Proposition 7.4. (W, w_∞) is a normal closure pair for (\mathbb{W}, d_0) .

Proposition 7.5. X has transfinite path products relative to $H \leq \pi_1(X, x_0)$ if and only if H is (W, w_∞) -closed.

Proof. One direction is straightforward. Suppose H is (W, w_∞) -closed, $\{0, 1\} \subseteq A \subseteq [0, 1]$ where A is closed and $p, q : ([0, 1], 0) \rightarrow (X, x_0)$ are paths such that $p|_A = q|_A$ and $[p|_{[0, d]} \cdot q|_{[c, d]}^- \cdot p|_{[0, c]}^-] \in H$ for every component (c, d) of $[0, 1] \setminus A$. By Remark 7.2, we may assume A is nowhere dense in $[0, 1]$. Find a non-decreasing, continuous, surjection $h : [0, 1] \rightarrow [0, 1]$ mapping the middle third cantor set \mathcal{C} onto A and such that every component of $[0, 1] \setminus \mathcal{C}$ is either mapped homeomorphically onto some component of $[0, 1] \setminus A$ or mapped to a point. Define $f : \mathbb{W} \rightarrow X$ so that $f \circ v_\infty = p \circ h$ and $f \circ \lambda_\infty = q \circ h$. Since $p \circ h$ and $q \circ h$ agree on \mathcal{C} , f is well-defined. Continuity at each point of $\mathcal{C} \times \{0\}$ follows directly from the continuity of p and q . Fix $(a, b) \in \mathcal{I}$ and let $k = [v_\infty|_{[0, a]} \cdot v_\infty|_{[a, b]} \cdot \lambda_\infty|_{[a, b]}^- \cdot v_\infty|_{[0, a]}^-]$ be the corresponding generator of W . If h maps (a, b) to a point, then $f_\#(k) = 1$. If h maps (a, b) onto a component (c, d) of $[0, 1] \setminus A$, then $f_\#(k) = [p|_{[0, c]}][p|_{[c, d]} \cdot q|_{[c, d]}^-][p|_{[0, c]}]^{-1} \in H$. This proves $f_\#(W) \leq H$ which allows us to conclude that $[p \circ q^-] = f_\#(w_\infty) \in H$. \square

Proposition 7.6. Let $H \leq \pi_1(X, x_0)$ be a subgroup.

- (1) If H is (D, d_∞) -closed, then H is (W, w_∞) -closed.
- (2) If H is (W, w_∞) -closed, then H is (C, c_τ) -closed.
- (3) If H is normal and (W, w_∞) -closed, then H is (P, p_τ) -closed.

Proof. (1) We view \mathbb{W} as a specific retract of \mathbb{D} and apply Remark 2.4. For a dyadic unital pair (n, j) , let $I_{n,j} = (\frac{j-1}{2^{n-1}}, \frac{j}{2^{n-1}})$. Consider the following recursively defined subset $A \subseteq \mathcal{D}$ of dyadic unital pairs: $A_3 = \{(3, 2)\}$ and

$$A_{n+2} = \{(n+2, 4j-2) \mid \text{for all } m < n \text{ and } (m, i) \in A_m, I_{n+2, 4j-2} \cap I_{m,i} = \emptyset\}.$$

Set $A = A_3 \cup A_5 \cup A_7 \cup \dots$. The intervals $\{I_{n,j} \mid (n, j) \in A\}$ are disjoint and have dense union in $[0, 1]$. Consequently, the subspace $\mathbb{W}' = B \cup \bigcup_{(n,j) \in A} \mathbb{D}(n, j) \subseteq \mathbb{D}$ is a homeomorphic copy of \mathbb{W} (see Figure 10). Let $v'_\infty(t) = (t, v(t))$ be the arc along $\bigcup_{(n,j) \in A} \mathbb{D}(n, j)$ from d_0 to $(1, 0)$. Let $s : \mathbb{W} \rightarrow \mathbb{W}'$ be a homeomorphism such that $s \circ \lambda_\infty \equiv \lambda_\infty$ and $s \circ v_\infty \equiv v'_\infty$. Define a retraction $r : \mathbb{D} \rightarrow \mathbb{W}'$ by downward vertical projection: Given $(t, y) \in \mathbb{D}$ if $y \geq v(t)$, let $r(t, y) = v'_\infty(t)$ and if $0 \leq y < v(t)$, let $r(t, y) = (t, 0)$. By construction, $r|_{\mathbb{W}'} = id_{\mathbb{W}'}$. Notice that $r_\#(d_\infty) = [v'_\infty \cdot (\lambda'_\infty)^-] = s_\#(w_\infty)$. Therefore, it suffices to show $r_\#(D) \leq s_\#(W)$. Recall that the free generator $d_{n,j}$ of D is the homotopy class of the loop $L_{n,j} = \left(\delta_{\frac{2j-1}{2^n}}\right) \cdot (\ell_{n+1, 2j}) \cdot \left(\delta_{\frac{j}{2^{n-1}}}\right)^-$.

For the moment, fix a dyadic rational $t \in [0, 1]$. Given the construction of the path δ_t , it is clear that part of the image of δ_t lies strictly below the image of v'_∞ if and only if $t \in U = \bigcup_{(n,j) \in A} I_{n,j}$. If $t \in I_{n,j} = (a, b)$, then $\delta_t \equiv \delta_a \cdot \zeta$ where $\zeta|_{(0,1]}$ lies strictly below the arc $\ell_{n,j}$. So, if $t \in U$, then $r \circ \delta_t \equiv v'_\infty|_{[0,a]} \cdot \lambda_\infty|_{[a,t]}$. On the other hand, if $t \notin U$, then δ_t has image either on or above the image of v'_∞ and $r \circ \delta_t \equiv v'_\infty|_{[0,t]}$.

Now fix a dyadic unital pair (n, j) . We claim that $r_\#(d_{n,j}) \in s_\#(W)$. There are three cases to consider:

Case I: Suppose $\ell_{n+1, 2j}$ has image on or above the image of v'_∞ . Then both $t = \frac{2j-1}{2^n} \notin U$ and $t' = \frac{j}{2^{n-1}} \notin U$. It follows that

$$\begin{aligned} r \circ L_{n,j} &= (r \circ \delta_t) \cdot (r \circ \ell_{n+1, 2j}) \cdot (r \circ \delta_{t'})^- \\ &\equiv v'_\infty|_{[0,t]} \cdot v'_\infty|_{[t,t']} \cdot v'_\infty|_{[0,t']}^- \end{aligned}$$

is null-homotopic. Thus $r_\#(d_{n,j}) = 1$.

Case II: Suppose $\ell_{n+1, 2j}$ has image strictly under the arc $\ell_{m,k}$ where $(m, k) \in A$ and $\ell_{n+1, 2j}(1) \neq \ell_{m,k}(1)$. Then both $t = \frac{2j-1}{2^n}$ and $t' = \frac{j}{2^{n-1}}$ lie in $I_{m,k} = (a, b)$. It follows that

$$\begin{aligned} r \circ L_{n,j} &= (r \circ \delta_t) \cdot (r \circ \ell_{n+1, 2j}) \cdot (r \circ \delta_{t'})^- \\ &\equiv (v'_\infty|_{[0,a]} \cdot \lambda_\infty|_{[a,t]}) \cdot (\lambda_\infty|_{[t,t']}) \cdot (v'_\infty|_{[0,a]} \cdot \lambda_\infty|_{[a,t']})^- \end{aligned}$$

is null-homotopic. Thus $r_\#(d_{n,j}) = 1$.

Case III: Suppose $\ell_{n+1, 2j}$ has image strictly under the arc $\ell_{m,k}$ where $(m, k) \in A$ and $\ell_{n+1, 2j}(1) = \ell_{m,k}(1)$. Then $t = \frac{2j-1}{2^n} \in I_{m,k} = (a, b)$ and $b = \frac{j}{2^{n-1}} \notin (a, b)$. Let $(c, d) \in \mathcal{I}$ such that $s([c, d]) = [a, b]$. It follows that

$$\begin{aligned} r \circ L_{n,j} &= (r \circ \delta_t) \cdot (r \circ \ell_{n+1, 2j}) \cdot (r \circ \delta_b)^- \\ &\equiv (v'_\infty|_{[0,a]} \cdot \lambda_\infty|_{[a,t]}) \cdot (\lambda_\infty|_{[t,b]}) \cdot (v'_\infty|_{[0,b]})^- \\ &\equiv s \circ (v_\infty|_{[0,c]} \cdot \lambda_\infty|_{[c,d]} \cdot v_\infty|_{[0,d]})^- \end{aligned}$$

where $[v_\infty|_{[0,c]} \cdot \lambda_\infty|_{[c,d]} \cdot v_\infty|_{[0,d]}^-]$ is the inverse of a generator of W . Thus $r_\#(d_{n,j}) \in s_\#(W)$.

(2) Define a map $f : (\mathbb{W}, d_0) \rightarrow (\mathbb{H}^+, b_0^+)$ so that $f \circ v_\infty|_{[0,2/3]} \equiv f \circ \lambda_\infty|_{[0,2/3]} \equiv \iota$, $f \circ v_\infty|_{[2/3,1]} \equiv \ell_\tau$ and $f \circ \lambda_\infty|_{[2/3,1]}$ is constant at b_0 . Since $f_\#(W) \leq C$ and $f_\#(w_\infty) = c_\tau$, we may apply Remark 2.4.

(3) Suppose H is a (W, w_∞) -closed, normal subgroup of $\pi_1(X, x_0)$ and $f : (\mathbb{H}^+, b_0^+) \rightarrow (X, x_0)$ is a map such that $f_\#(P) \leq H$. Let $\alpha = f \circ \iota$ and recall that H^α is (W, w_∞) -closed. Define $g : (\mathbb{W}, d_0) \rightarrow (X, f(b_0))$ so that $g(t, 0) = f(b_0)$ if $t \in \mathcal{C}$, $g \circ \lambda_\infty = f \circ f_{\text{odd}} \circ \ell_\tau$, and $g \circ v_\infty = f \circ f_{\text{even}} \circ \ell_\tau$. We have $g_\#([v_\infty|_{[a,b]} \cdot \lambda_\infty|_{[a,b]}^-]) \in H^\alpha$ for each $(a, b) \in \mathcal{I}$. Since H^α is normal, $g_\#(W) \leq H^\alpha$. By assumption, we now have $g_\#(w_\infty) \in H^\alpha$. Thus $f_\#(p_\tau) = [\alpha]g_\#(w_\infty)[\alpha^-] \in H$. \square

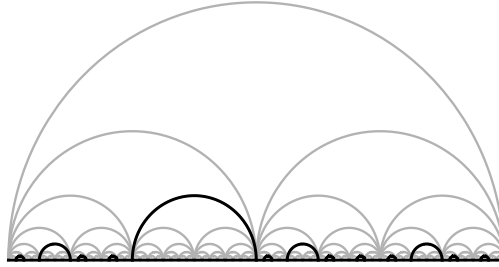


FIGURE 10. The subspace \mathbb{W}' of \mathbb{D} .

We prove a partial converse to (1) of Proposition 7.6. To do so we require two technical lemmas. Let $\mathbf{w}(X) \subseteq X$ denote the “wild” subspace of points at which X is not semilocally simply-connected.

Proposition 7.7. *If X is any space, Y is locally path connected and $f : Y \rightarrow X$ is continuous, then $f^{-1}(\mathbf{w}(X))$ is closed in Y . In particular, if X is locally path connected, then $\mathbf{w}(X)$ is closed in X .*

Proof. Pick $y \notin f^{-1}(\mathbf{w}(X))$. There is an open neighborhood U of $f(y)$ such the inclusion $U \rightarrow X$ induces the trivial homomorphism $\pi_1(U, f(y)) \rightarrow \pi_1(X, f(y))$. If C is the path component of $f(y)$ in U , then the inclusion $C \rightarrow X$ also induces the trivial homomorphism $\pi_1(C, f(y)) \rightarrow \pi_1(X, f(y))$. Find a path-connected neighborhood V of y such that $f(V) \subseteq U$. If $v \in V$, then $f(v) \in C$. Therefore the inclusion $U \rightarrow X$ induces the trivial homomorphism $\pi_1(U, f(v)) \rightarrow \pi_1(X, f(v))$. Thus $V \cap f^{-1}(\mathbf{w}(X)) = \emptyset$. \square

Lemma 7.8. *Let $H \leq \pi_1(X, x_0)$ be a subgroup.*

- (1) *If H is (C, c_∞) -closed, $\alpha \in P(X, x_0)$, and $f : \mathbb{D} \rightarrow X$ is a map such that $f(B) = \alpha(1)$ and $f_\#(S) \leq H^\alpha$, then $f_\#(d_\infty) \in H^\alpha$.*
- (2) *If $\alpha \in P(X, x_0)$ and $f : (\mathbb{D}, d_0) \rightarrow (X, \alpha(1))$ is a map such that $f(B) \subseteq X \setminus \mathbf{w}(X)$ and $f_\#(S) \leq H^\alpha$, then $f_\#(d_\infty) \in H^\alpha$.*

Proof. (1) Since f maps B to a point, the loops $f \circ \lambda_n$ are null at $\alpha(1)$. Define a map $f' : (\mathbb{H}^+, b_0^+) \rightarrow (X, x_0)$ such that $f' \circ \iota = \alpha$ and $f' \circ \ell_n = f \circ (\lambda_n \cdot \lambda_{n+1}^-)$ for

$n \in \mathbb{N}$. Since $(f')_{\#}(C)^{\alpha} = (f)_{\#}(S) \leq H^{\alpha}$, we have $(f')_{\#}(C) \leq H$ and since H is (C, c_{∞}) -closed, we have $(f')_{\#}(c_{\infty}) \in H$. But

$$(f')_{\#}(c_{\infty}) = [\alpha] \left[\left(\prod_{n=1}^{\infty} (f \circ (\lambda_n \cdot \lambda_{n+1}^{-})) \right) \right] [\alpha^{-}] = [\alpha \cdot (f \circ \lambda_1) \cdot \alpha^{-}] = [\alpha] f_{\#}(d_{\infty}) [\alpha^{-}].$$

Thus $f_{\#}(d_{\infty}) \in H^{\alpha}$.

(2) For every $0 \leq t \leq 1$, choose an open neighborhood U_t of $f((t, 0))$ such that every loop in the path component of $f((t, 0))$ in U_t is null-homotopic in X . Find a path-connected open set W_t in \mathbb{D} such that $(t, 0) \in W_t \subseteq f^{-1}(U_t)$. Recall from Proposition 5.3 that $\mathbb{D}_{n,j}$ is the homeomorphic copy of \mathbb{D} beneath the arc $\ell_{n,j}$. There exists $n \geq 1$ such that for each $j = 1, 2, \dots, 2^{n-1}$, we have $\mathbb{D}_{n,j} \subseteq W_{t_j}$ for some t_j . Note $f(\mathbb{D}_{n,j})$ and $f((t_j, 0))$ lie in the same path component of U_{t_j} so if ζ_j is the simple closed curve traversing the outer curve of $\mathbb{D}_{n,j}$ in the counterclockwise direction, then $f \circ \zeta_j$ is null-homotopic in X . It follows that $f \circ (\lambda_n \cdot \lambda_{\infty}^{-})$ is null-homotopic in X . Thus $f_{\#}(d_{\infty}) = [f \circ (\lambda_1 \cdot \lambda_n^{-})][f \circ (\lambda_n \cdot \lambda_{\infty}^{-})] = [f \circ (\lambda_1 \cdot \lambda_n^{-})] \in f_{\#}(S) \leq H^{\alpha}$. \square

Lemma 7.9. *Suppose $N \trianglelefteq \pi_1(X, x_0)$ is a normal (W, w_{∞}) -closed subgroup. Let $f : (\mathbb{D}, d_0) \rightarrow (X, x_0)$ be a map such that $f_{\#}(D) \leq N$. Let $\mathcal{J} \subseteq \mathcal{D}$ be a collection of dyadic unital pairs (n, j) so that the corresponding intervals $I_{n,j} = (\frac{j-1}{2^{n-1}}, \frac{j}{2^{n-1}})$ satisfy:*

- (1) *if $(n_1, j_1), (n_2, j_2) \in \mathcal{J}$ and $(n_1, j_1) \neq (n_2, j_2)$, then $I_{n_1, j_1} \cap I_{n_2, j_2} = \emptyset$,*
- (2) *$U = \bigcup_{(n,j) \in \mathcal{J}} I_{n,j}$ is dense in $[0, 1]$.*

Let $s : [0, 1] \rightarrow \mathbb{D}$ be the path defined as $s(t) = (t, 0)$ if $t \in [0, 1] \setminus U$ and $s|_{\overline{I_{n,j}}} \equiv \ell_{n,j}$ if $(n, j) \in \mathcal{J}$. Then $f_{\#}([\lambda_1 \cdot s^{-}]) \in N$.

Proof. The lemma is clear if \mathcal{J} is finite. Assume \mathcal{J} is infinite. Then $(1, 1) \notin \mathcal{J}$. We define a path $\gamma : [0, 1] \rightarrow \mathbb{D}$, which is homotopic to λ_1 . If $t \in [0, 1] \setminus U$, set $\gamma(t) = (t, 0)$. On the intervals $\overline{I_{n,j}}$, $(n, j) \in \mathcal{J}$, we define γ to be a path $\gamma_{n,j}$, which is a finite concatenation of standard paths from $\ell_{n,j}(0)$ to $\ell_{n,j}(1)$, using the following inductive procedure:

First, if $a = \frac{j}{2^{n-1}} \leq \frac{j'}{2^{n-1}} = b$ are dyadic rationals, let $\Lambda_n(a, b)$ denote the arc $\prod_{i=j+1}^{j'} \ell_{n,i}$ on the n -th level from $(a, 0)$ to $(b, 0)$. In the case that $a = b$, $\Lambda_n(a, b)$ is the constant path.

To begin the induction, put $J_1 = \{(1, 1)\}$. Inductively, assume that J_q has been defined as a nonempty, finite set of dyadic unital pairs disjoint from \mathcal{J} . Let (m, p) be the smallest (in the dictionary order) element of J_q . By Assumption (2), there is a minimal $n > m$ such that there is a j with the property that $(n, j) \in \mathcal{J}$ and $I_{n,j} \subset I_{m,p}$; here, $I_{m,p}$ is defined analogous to $I_{n,j}$. Let $j_1 < j_2 < \dots < j_r$ be the complete list of all such j s. Let $k_1 < k_2 < \dots < k_u$ be the complete (but possibly empty) list of all k 's such that $(n, k_i) \notin \mathcal{J}$ and $\bigcup_i I_{n,k_i} \cup \bigcup_i I_{n,j_i} = \overline{I_{m,p}}$. Define

$$\gamma_{n,j_1} = \left(\Lambda_n \left(\frac{p-1}{2^{m-1}}, \frac{j_1-1}{2^{n-1}} \right) \right)^{-} \cdot \ell_{m,p} \cdot \left(\Lambda_n \left(\frac{j_1}{2^{n-1}}, \frac{p}{2^{m-1}} \right) \right)^{-}$$

(see Figure 11). For $i > 1$, define $\gamma_{n,j_i} = \ell_{n,j_i}$. Define J_{q+1} from J_q by removing (m, p) and adding $(n, k_1), (n, k_2), \dots, (n, k_u)$. Since \mathcal{J} is infinite, J_{q+1} is guaranteed to be nonempty even if no k_i exist. This completes the induction.

By Assumption (1), $\gamma_{n,j}$ has now been defined for every $(n,j) \in \mathcal{J}$. Hence, γ has been defined. Note that γ is uniformly continuous, because for every $\epsilon > 0$, only finitely many $\mathbb{D}_{m,p}$ have diameter $> \epsilon$.

Consider the retraction $r_n : \mathbb{D} \rightarrow E_n$ from the introduction to Section 4. Observe that for $(m,p) \in J_q$ as in the above induction, we have $r_m(\gamma(I_{m,p} \setminus I_{n,j_1})) \subseteq B$. Since the corresponding statement also applies to the elements $(n,k_i) \in J_{q+1}$, we have that $r_n \circ \gamma$ restricted to $\overline{I_{m,p}}$ is homotopic to $\ell_{m,p}$. We conclude that, for all $n \in \mathbb{N}$, $r_n \circ \gamma$ is homotopic to $r_n \circ \lambda_1 = \lambda_1$ and thus $\gamma \simeq \lambda_1$.

Since $[\gamma_{n,j} \cdot \ell_{n,j}^-] \in D$ for each $(n,j) \in \mathcal{J}$ and N is normal, $f_{\#}([\alpha \cdot \gamma_{n,j} \cdot \ell_{n,j}^- \cdot \alpha^-]) \in N$ for every path $\alpha : [0,1] \rightarrow \mathbb{D}$ from d_0 to $\ell_{n,j}(0)$. Therefore, the paths s and γ agree on $[0,1] \setminus U$ and for each component $(a,b) = I_{n,j}$ of U , we have

$$f_{\#}([\gamma|_{[0,b]} \cdot s|_{[a,b]} \cdot \gamma|_{[0,a]}]) = f_{\#}([\gamma|_{[0,a]} \cdot (\gamma_{n,j} \cdot \ell_{n,j}^-) \cdot \gamma|_{[0,a]}]) \in N.$$

Since X is assumed to have transfinite path products relative to N , we conclude that $f_{\#}([\lambda_1 \cdot s^-]) = f_{\#}([\gamma \cdot s^-]) \in N$. \square

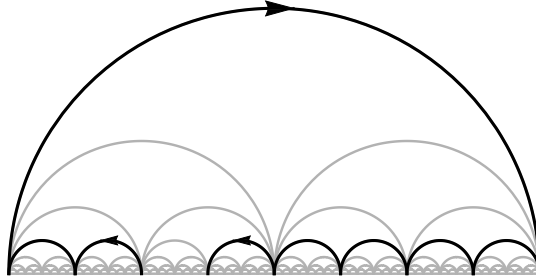


FIGURE 11. An example of the definition of γ_{n,j_1} in $\mathbb{D}_{m,p}$ when $n = m + 3$ and $j_1 = 2^{n-m}(p-1) + 3$.

Theorem 7.10. *Suppose $\mathbf{w}(X)$ is totally path disconnected and $N \leq \pi_1(X, x_0)$ is a normal subgroup. Then N is (D, d_∞) -closed if and only if N is (W, w_∞) -closed. In particular, the closure operators cl_{D, d_∞} and cl_{W, w_∞} agree on the normal subgroups of $\pi_1(X, x_0)$.*

Proof. One direction follows from Proposition 7.6. For the other direction, suppose X has transfinite path products relative to N and $f : (\mathbb{D}, d_0) \rightarrow (X, x_0)$ is a map such that $f_{\#}(D) \leq N$. If $f(B) = x_0$ or $f(B) \subseteq X \setminus \mathbf{w}(X)$, then we may apply Lemma 7.8 (recall that $S \leq D$) to conclude that $f_{\#}(d_\infty) \in N$. Thus we may assume that $f|_B$ is nonconstant and has image intersecting $\mathbf{w}(X)$. By Proposition 7.7, $Y = f^{-1}(X \setminus \mathbf{w}(X)) \cap ((0,1) \times \{0\})$ is open in $(0,1) \times \{0\}$. Let Z be the (possibly empty) interior of $((0,1) \times \{0\}) \setminus Y$ in $(0,1) \times \{0\}$. Note that $V = Y \cup Z$ is open and dense in B .

Since $\mathbf{w}(X)$ is totally path disconnected and $f(Z) \subseteq \mathbf{w}(X)$, each connected component of Z must be mapped by f to a single point. Let \mathcal{J}_Y be a collection of dyadic unital pairs (n,j) such that the union of the corresponding intervals $I_{n,j} = (\frac{j-1}{2^{n-1}}, \frac{j}{2^{n-1}})$ are disjoint and dense in Y . Similarly, let \mathcal{J}_Z be a (possibly empty) collection of dyadic unital pairs (n,j) such that the union of the corresponding intervals $I_{n,j}$ are disjoint and dense in Z . Let $\mathcal{J} = \mathcal{J}_Y \cup \mathcal{J}_Z$ and $U = \bigcup_{(n,j) \in \mathcal{J}} I_{n,j} \subseteq$

V . Note that both Conditions (1) and (2) in Lemma 7.9 are satisfied so the path $s : [0, 1] \rightarrow \mathbb{D}$ as defined in the statement of the Lemma satisfies $f_{\#}([\lambda_1 \cdot s^-]) \in N$.

Fix a dyadic unital pair $(n, j) \in \mathcal{J}$ and put $(a, b) = I_{n,j}$. Consider $\mathbb{D}_{n,j} \subseteq \mathbb{D}$ and the homeomorphism $T_{n,j} : \mathbb{D} \rightarrow \mathbb{D}_{n,j}$ as defined in Proposition 5.3. Set $f_{n,j} = f \circ T_{n,j}$. Let $\beta_{n,j} : [0, 1] \rightarrow \mathbb{D}$ be the path which is the restriction of λ_{∞} to $\overline{I_{n,j}}$. Since N is normal and $f_{\#}(D) \leq N$, we have $(f_{n,j})_{\#}(S) \leq (f_{n,j})_{\#}(D) \leq N^{\alpha}$ for every path $\alpha : [0, 1] \rightarrow X$ from x_0 to $f(a, 0)$. Notice that

- (1) $f_{n,j}(B) \subseteq X \setminus \mathbf{w}(X)$ if $(n, j) \in \mathcal{J}_Y$,
- (2) $f_{n,j}(B)$ is a single point if $(n, j) \in \mathcal{J}_Z$.

In either case, we may apply Lemma 7.8 to see that $(f_{n,j})_{\#}(d_{\infty}) = [f \circ (\ell_{n,j} \cdot \beta_{n,j}^-)] \in N^{\alpha}$ for $\alpha = f \circ s|_{[0,a]}$.

By construction, the paths s and λ_{∞} agree on $[0, 1] \setminus U$. Moreover, for every component $(a, b) = I_{n,j}$ of U , we have

$$f_{\#}([s|_{[0,b]} \cdot \lambda_{\infty}|_{[a,b]}^- \cdot s|_{[0,a]}^-]) = f_{\#}([s|_{[0,a]} \cdot \ell_{n,j} \cdot \beta_{n,j}^- \cdot s|_{[0,a]}^-]) \in N.$$

Since X is assumed to have transfinite products relative to N , we conclude that $f_{\#}([s \circ \lambda_{\infty}^-]) \in N$. Thus $f_{\#}(d_{\infty}) = f_{\#}([\lambda_1 \cdot s^-])f_{\#}([s \cdot \lambda_{\infty}^-]) \in N$. The last statement of the theorem follows from Corollary 2.7. \square

We conclude this paper by considering spaces X with a discrete wild set $\mathbf{w}(X)$.

Lemma 7.11. *Suppose $H \leq \pi_1(X, x_0)$ is (C, c_{∞}) -closed, $\gamma \in P(X, x_0)$, and A is a nowhere dense closed subset of $[0, 1]$ containing $\{0, 1\}$. If $\alpha, \beta \in P(X, \gamma(1))$ are paths such that*

- (1) $\alpha|_A = \beta|_A$,
- (2) $[\alpha|_{[0,b]} \cdot \beta|_{[a,b]}^- \cdot \alpha|_{[0,a]}^-] \in H^{\gamma}$ for all components (a, b) of $[0, 1] \setminus A$,
- (3) $\alpha((0, 1)) \cap \mathbf{w}(X) = \emptyset$ and $\beta((0, 1)) \cap \mathbf{w}(X) = \emptyset$,

then $[\alpha \cdot \beta^-] \in H^{\gamma}$.

Proof. By the proof of Proposition 7.5, we may reparameterize α, β and construct a map $f : \mathbb{W} \rightarrow X$ such that $f \circ v_{\infty} = \alpha$, $f \circ \lambda_{\infty} = \beta$ and $f_{\#}(W) \leq H^{\gamma}$. It suffices to show that $[\alpha \cdot \beta^-] = f_{\#}(w_{\infty}) \in H^{\gamma}$. Let $x_1 = f(d_0)$ and $x_2 = f(1, 0)$. We consider four cases.

Case I: Suppose $x_1 \notin \mathbf{w}(X)$ and $x_2 \notin \mathbf{w}(X)$. Then $f(\mathbb{W}) \subseteq X \setminus \mathbf{w}(X)$. Recall from the proof of Proposition 7.6 (1) that there is a map $g : (\mathbb{D}, d_0) \rightarrow (\mathbb{W}, d_0)$ such that $g|_B = id_B$, $g_{\#}(D) \leq W$ and $g_{\#}(d_{\infty}) = w_{\infty}$. Therefore $f \circ g : (\mathbb{D}, d_0) \rightarrow (X, \alpha(1))$ is a map such that $f \circ g(B) \subseteq X \setminus \mathbf{w}(X)$ and $(f \circ g)_{\#}(S) \leq (f \circ g)_{\#}(D) \leq f_{\#}(W) \leq H^{\gamma}$. By (2) of Lemma 7.8, we have $f_{\#}(w_{\infty}) = (f \circ g)_{\#}(d_{\infty}) \in H^{\gamma}$.

Case II: Suppose $x_1 \in \mathbf{w}(X)$ and $x_2 \notin \mathbf{w}(X)$. Define $(t_n)_{n \geq 1}$ to be the sequence in the Cantor set \mathcal{C} given by $t_{2m-1} = \frac{1}{3^{m-1}}$ and $t_{2m} = \frac{2}{3^m}$. Note that $\{(x, y) \in \mathbb{W} | t_{n+1} \leq x \leq t_n\}$ is homeomorphic to either \mathbb{W} when n is odd or S^1 when n is even. Since $\alpha((0, 1)) \cap \mathbf{w}(X) = \emptyset$ and $\beta((0, 1)) \cap \mathbf{w}(X) = \emptyset$, we have $[\alpha|_{[0,t_n]} \cdot \beta|_{[t_{n+1}, t_n]}^- \cdot \alpha|_{[0,t_{n+1}]}^-] \in H^{\gamma}$ by assumption when n is even and $[\alpha|_{[t_{n+1}, t_n]} \cdot \beta|_{[t_{n+1}, t_n]}^-] \in H^{\gamma \cdot \alpha|_{[0,t_{n+1}]}}$ by Case I when n is odd. Thus $[\alpha|_{[0,t_n]} \cdot \beta|_{[t_{n+1}, t_n]}^- \cdot \alpha|_{[0,t_{n+1}]}^-] \in H^{\gamma}$ for all $n \in \mathbb{N}$. Define a map $k : \mathbb{H}^+ \rightarrow X$ by $k \circ \iota = \gamma$, and $k \circ \ell_n = \alpha|_{[0,t_n]} \cdot \beta|_{[t_{n+1}, t_n]}^- \cdot \alpha|_{[0,t_{n+1}]}^-$. Since $k_{\#}(C) \leq H$ and H is (C, c_{∞}) -closed, we have $k_{\#}(c_{\infty}) \in H$. Thus

$[k \circ \ell_\infty] \in H^\gamma$. However,

$$\begin{aligned}
[k \circ \ell_\infty] &= \left[\prod_{n=1}^{\infty} \left(\alpha|_{[0,t_n]} \cdot \beta|_{[t_{n+1},t_n]}^- \cdot \alpha|_{[0,t_{n+1}]}^- \right) \right] \\
&= [\alpha|_{[0,t_1]}] \left[\prod_{n=1}^{\infty} \left(\beta|_{[t_{n+1},t_n]}^- \cdot \alpha|_{[0,t_{n+1}]}^- \cdot \alpha|_{[0,t_{n+1}]} \right) \right] \\
&= [\alpha|_{[0,t_1]}] \left[\prod_{n=1}^{\infty} \beta|_{[t_{n+1},t_n]}^- \right] \\
&= [\alpha \cdot \beta^-]
\end{aligned}$$

Case III: Suppose $x_1 \notin \mathbf{w}(X)$ and $x_2 \in \mathbf{w}(X)$. Define $f' : \mathbb{W} \rightarrow X$ by $f'(x, y) = f(1-x, y)$. Since $(f')_\#(W) \leq H^{\gamma \cdot \alpha}$, we may use Case II to conclude that $(f')_\#(w_\infty) = [\alpha^- \cdot \beta] \in H^{\gamma \cdot \alpha}$. Conjugating by $[\alpha]$ and inverting gives $[\alpha \cdot \beta^-] \in H^\gamma$.

Case IV: Suppose $x_1 \in \mathbf{w}(X)$ and $x_2 \in \mathbf{w}(X)$. Define maps $f_1, f_2 : \mathbb{W} \rightarrow X$ so that $f_1 \circ v_\infty \equiv \alpha|_{[0,1/3]}$, $f_1 \circ \lambda_\infty \equiv \beta|_{[0,1/3]}$, $f_2 \circ v_\infty \equiv \alpha|_{[2/3,1]}$, and $f_2 \circ \lambda_\infty \equiv \beta|_{[2/3,1]}$. Applying Case II to f_1 , we see that $[\alpha|_{[0,1/3]} \cdot \beta|_{[0,1/3]}^-] \in H^\gamma$. Applying Case III to f_2 , we see that $[\alpha|_{[2/3,1]} \cdot \beta|_{[2/3,1]}^-] \in H^{\gamma \cdot \alpha|_{[0,2/3]}}$. Thus $[\alpha \cdot \beta|_{[2/3,1]}^- \cdot \alpha|_{[0,2/3]}^-] \in H^\gamma$. By assumption, we have $[\alpha|_{[0,2/3]} \cdot \beta|_{[1/3,2/3]}^- \cdot \alpha|_{[0,1/3]}^-] \in H^\gamma$. It follows that

$$[\alpha \cdot \beta^-] = [\alpha \cdot \beta|_{[2/3,1]}^- \cdot \alpha|_{[0,2/3]}^-] [\alpha|_{[0,2/3]} \cdot \beta|_{[1/3,2/3]}^- \cdot \alpha|_{[0,1/3]}^-] [\alpha|_{[0,1/3]} \cdot \beta|_{[0,1/3]}^-] \in H^\gamma$$

□

Lemma 7.12. *Suppose $N \trianglelefteq \pi_1(X, x_0)$ is a normal (P, p_τ) -closed subgroup, $\gamma \in P(X, x_0)$, and A is a nowhere dense closed subset of $[0, 1]$ containing $\{0, 1\}$. If $\alpha, \beta \in P(X, \gamma(1))$ are paths such that*

- (1) $\alpha|_A = \beta|_A$,
- (2) $[\alpha|_{[0,b]} \cdot \beta|_{[a,b]}^- \cdot \alpha|_{[0,a]}^-] \in N^\gamma$ for every component (a, b) of $[0, 1] \setminus A$,
- (3) *there is a point $x_1 \in X$ such that $x_1 \in \alpha([a, b]) \cup \beta([a, b])$ for all components (a, b) of $[0, 1] \setminus A$,*

then $[\alpha \cdot \beta^-] \in N^\gamma$.

Proof. Let \mathcal{C} denote the set of components of $[0, 1] \setminus A$. First, we note that if \mathcal{C} is finite, then the conclusion is clear since, in this case, $[\alpha \cdot \beta^-]$ factors as a product of the elements $[\alpha|_{[0,b]} \cdot \beta|_{[a,b]}^- \cdot \alpha|_{[0,a]}^-] \in N^\gamma$, $(a, b) \in \mathcal{C}$. Therefore, we assume A and \mathcal{C} are infinite. It follows from assumption (3) that if a is a limit point of A , then there exists a sequence $t_n \rightarrow a$ in $[0, 1]$ such that either $\alpha(t_n) = x_1$ or $\beta(t_n) = x_1$ for all n . Hence $\alpha(a) = x_1$ for all limits points a of A . Additionally, since A is compact, if U is any open neighborhood of x_1 , then we must have $\alpha([a, b]) \cup \beta([a, b]) \subseteq U$ for all but finitely many $(a, b) \in \mathcal{C}$.

For each point $a \in A$, we define a path ρ_a from x_1 to $\alpha(a) = \beta(a)$. If a is a limit point of A , let $\rho_a : \{a\} \rightarrow X$ be the degenerate constant path at x_1 . If $a \in A \cap [0, 1]$ is an isolated point, there is a $b \in A$ such that $(a, b) \in \mathcal{C}$. If there exists a smallest $s \in [a, b]$ such that $\alpha(s) = x_1$, define $\rho_a \equiv \alpha|_{[a,s]}^-$. If no such s exists, then there exists a smallest $s \in [a, b]$ such that $\beta(s) = x_1$; in this case set $\rho_a \equiv \beta|_{[a,s]}^-$. If 1 is an isolated point of A , take ρ_1 to be any path from x_1 to $\alpha(1) = \beta(1)$. Define loops $L, M : [0, 1] \rightarrow X$ so that $L(A) = M(A) = x_1$ and if (a, b) is a component of $[0, 1] \setminus A$, set $L|_{[a,b]} \equiv \rho_a \cdot \alpha|_{[a,b]} \cdot \rho_b^-$ and $M|_{[a,b]} \equiv \rho_a \cdot \beta|_{[a,b]} \cdot \rho_b^-$. Note that for any

open neighborhood U of x_1 , all but finitely many $L|_{[a,b]}$ and $M|_{[a,b]}$ have image in U . It follows that L and M are continuous.

The loops L and M are constructed from α and β respectively by inserting an at most countably infinite number of loops $\rho_a \cdot \rho_a^-$ at the isolated points $a \in A$ with $0 < a < 1$, prepending ρ_0 and appending ρ_1^- . Each such loop contracts in it's own image. Since all but finitely many of these contractions lie within a given neighborhood of x_1 , there are homotopies $L \simeq \rho_0 \cdot \alpha \cdot \rho_1^-$ and $M \simeq \rho_0 \cdot \beta \cdot \rho_1^-$. Thus $[\alpha \cdot \beta^-] = [\rho_0^-][L \cdot M^-][\rho_0]$. We now seek to show $[L \cdot M^-] \in N^{\gamma \cdot \rho_0^-}$.

Fix a component (a, b) of $[0, 1] \setminus A$ and recall $[\alpha|_{[0,b]} \cdot \beta|_{[a,b]}^- \cdot \alpha|_{[0,a]}^-] \in N^\gamma$. Conjugating by $[\alpha|_{[0,a]}^-]$ gives $[\alpha|_{[a,b]} \cdot \beta|_{[a,b]}^-] = [\alpha|_{[0,a]}^-][\alpha|_{[0,b]} \cdot \beta|_{[a,b]}^- \cdot \alpha|_{[0,a]}^-][\alpha|_{[0,a]}] \in N^{\gamma \cdot \alpha|_{[0,a]}}$. Thus

$$[L|_{[a,b]} \cdot M|_{[a,b]}^-] = [\rho_a][\alpha|_{[a,b]} \cdot \beta|_{[a,b]}^-][\rho_a^-] \in N^{\gamma \cdot \alpha|_{[0,a]} \cdot \rho_a^-} = N^{\gamma \cdot \rho_0^-}$$

where the equality $N^{\gamma \cdot \alpha|_{[0,a]} \cdot \rho_a^-} = N^{\gamma \cdot \rho_0^-}$ follows from the normality of N .

Enumerate the infinitely many components of $[0, 1] \setminus A$ as $(a_1, b_1), (a_2, b_2), \dots$. The loop L induces a map $f_\alpha : \mathbb{H}^+ \rightarrow X$ such that $f_\alpha \circ \iota = \gamma \cdot \rho_0^-$ and $f|_\alpha \circ \ell_n \equiv L|_{[a_n, b_n]}$. Similarly, M induces a map $f_\beta : \mathbb{H}^+ \rightarrow X$ such that $f_\beta \circ \iota = \gamma \cdot \rho_0^-$ and $f|_\beta \circ \ell_n \equiv M|_{[a_n, b_n]}$. Note that there is a loop ζ in \mathbb{H} based at b_0 such that $f_\alpha \circ \zeta \equiv L$ and $f_\beta \circ \zeta \equiv M$. Since \mathbb{H} is assumed to have transfinite products rel. N (recall Proposition 3.18) and $(f_\alpha)_\#(c_n)(f_\beta)_\#(c_n)^{-1} = [\gamma][\rho_0^-][L|_{[a_n, b_n]} \cdot M|_{[a_n, b_n]}^-][\rho_0][\gamma^-] \in N$ for each $n \in \mathbb{N}$, we have $[\gamma \cdot \rho_0^-][L \cdot M^-][\rho_0 \cdot \gamma^-] = (f_\alpha)_\#([\iota \cdot \zeta \cdot \iota])(f_\beta)_\#([\iota \cdot \zeta \cdot \iota])^{-1} \in N$, completing the proof. \square

Theorem 7.13. *Suppose $\mathbf{w}(X)$ is discrete and $N \trianglelefteq \pi_1(X, x_0)$ is a normal subgroup. Then N is (D, d_∞) -closed if and only if N is (P, p_τ) -closed. In particular, the closure operators cl_{D, d_∞} and cl_{P, p_τ} agree on the normal subgroups of $\pi_1(X, x_0)$.*

Corollary 7.14. *Suppose X is a metric space such that $\mathbf{w}(X)$ is discrete and $N \trianglelefteq \pi_1(X, x_0)$ is a normal subgroup. Then $p_N : \tilde{X}_N \rightarrow X$ has the unique path lifting property if and only if X has transfinite products relative to N . In particular, $p_K : \tilde{X}_K \rightarrow X$ has the unique path lifting property if $K = cl_{P, p_\tau}(N)$.*

Corollary 7.15. *Suppose X is a metric space such that $\mathbf{w}(X)$ is discrete and $N \leq \pi_1(X, x_0)$ contains the commutator subgroup of $\pi_1(X, x_0)$. Then $p_N : \tilde{X}_N \rightarrow X$ has the unique path lifting property if and only if X is homotopically Hausdorff relative to N . In particular, $p_K : \tilde{X}_K \rightarrow X$ has the unique path lifting property if $K = cl_{C, c_\infty}(N)$.*

Proof of Theorem 7.13. The last statement of the theorem follows from Corollary 2.7 once the equivalence is proven. One direction follows from Proposition 7.6. Suppose the normal subgroup $N \trianglelefteq \pi_1(X, x_0)$ is (P, p_τ) -closed. By Theorem 7.10, it suffices to show N is (W, w_∞) -closed. Let $f : (\mathbb{W}, d_0) \rightarrow (X, x_0)$ be a map such that $f_\#(W) \leq N$. Denote $\alpha = f \circ v_\infty$ and $\beta = f \circ \lambda_\infty$ and observe $\alpha|_C = \beta|_C$ where C is the Cantor set. We seek to show that $[\alpha \cdot \beta^-] \in N$.

Our claim is trivial if $\mathbf{w}(X) = \emptyset$. Since (W, w_∞) is a normal closure pair, we are free to change the basepoint so that $x_0 \in \mathbf{w}(X)$. If $\alpha(1) = \beta(1) \neq x_0$, find a path γ from x_0 to $\alpha(1)$. Using the self-similarity of \mathbb{W} , we may replace f with a map $g : \mathbb{W} \rightarrow \mathbb{H}$ such that $g \circ v_\infty|_{[0, 1/3]} \equiv \alpha$, $g \circ \lambda_\infty|_{[0, 1/3]} \equiv \beta$ and $g \circ v_\infty|_{[1/3, 1]} = g \circ \lambda_\infty|_{[1/3, 1]} \equiv \gamma$. Clearly $g_\#(W) = f_\#(W)$ and $g_\#(w_\infty) = f_\#(w_\infty)$.

Therefore, without loss of generality, we may assume α, β are loops in X based at x_0 .

Since \mathbb{W} is a Peano continuum, $f^{-1}(\mathbf{w}(X))$ is closed in \mathbb{W} by Proposition 7.7 and therefore is compact. The continuous image of a compact set in a discrete space is finite. Therefore, we may list the points of $f(\mathbb{W}) \cap \mathbf{w}(X)$ as the finite set $\Omega = \{x_0, x_1, \dots, x_n\}$. Recall that \mathcal{I} denotes the set of components of $[0, 1] \setminus \mathcal{C}$. Let

$$\begin{aligned} A_0 &= \{t \in \mathcal{C} \mid f(t, 0) \in \Omega\}, \\ A_1 &= \{a \in \mathcal{C} \mid (a, b) \in \mathcal{I} \text{ and } (\alpha([a, b]) \cup \beta([a, b])) \cap \Omega \neq \emptyset\}, \\ A_2 &= \{b \in \mathcal{C} \mid (a, b) \in \mathcal{I} \text{ and } (\alpha([a, b]) \cup \beta([a, b])) \cap \Omega \neq \emptyset\}, \end{aligned}$$

and $A = A_0 \cup A_1 \cup A_2$. Certainly, A is nowhere dense in $[0, 1]$; we check that A is closed in $[0, 1]$ by showing A is closed in \mathcal{C} . Choose a point $t \in \mathcal{C} \setminus A$. Notice that $t \notin \alpha^{-1}(\Omega) \cup \beta^{-1}(\Omega)$. If $t = a$ for some $(a, b) \in \mathcal{I}$, then it must be the case that $(\alpha([a, b]) \cup \beta([a, b])) \cap \Omega = \emptyset$. Thus $[a, b] \cap A = \emptyset$. Since a does not lie in the closed set $\alpha^{-1}(\Omega) \cup \beta^{-1}(\Omega)$, there is a $c \in \mathcal{C}$ such that $c < a$ and $(c, a] \cap (\alpha^{-1}(\Omega) \cup \beta^{-1}(\Omega)) = \emptyset$. Thus $a \in (c, b)$ and by the definition of A , we have $(c, b) \cap A = \emptyset$. Similarly, if $t = b$ for some $(a, b) \in \mathcal{I}$, we may find a $c \in \mathcal{C}$ with $b < c$ such that $(a, c) \cap A = \emptyset$. Finally, if t is not an endpoint of any element of \mathcal{I} , then we may find $c, c' \in \mathcal{C}$ with $c < t < c'$ such that $(c, c') \cap (\alpha^{-1}(\Omega) \cup \beta^{-1}(\Omega)) = \emptyset$. Again, by the definition of A , we have $(c, c') \cap A = \emptyset$, finishing the proof that A is closed.

The open set $[0, 1] \setminus A$ is the disjoint union of open intervals (r, s) , each of which is either equal to some $(a, b) \in \mathcal{I}$ or to the union of infinitely many $(a, b) \in \mathcal{I}$ and points $t \in \mathcal{C} \setminus A$. We further classify the components (r, s) of $[0, 1] \setminus A$ as follows:

- (1) (r, s) is Type I if $(r, s) \in \mathcal{I}$.
- (2) (r, s) is Type II if $(r, s) \notin \mathcal{I}$ and $\Omega \cap \{\alpha(r), \alpha(s)\} \neq \emptyset$.
- (3) (r, s) is Type III if $(r, s) \notin \mathcal{I}$ and $\Omega \cap \{\alpha(r), \alpha(s)\} = \emptyset$.

If (r, s) is Type I, then $\Omega \cap (\alpha([r, s]) \cup \beta([r, s])) \neq \emptyset$ and $[\alpha|_{[0, s]} \cdot \beta|_{[r, s]}^- \cdot \alpha|_{[0, r]}^-] \in N$ by assumption. If (r, s) is Type II or III, then $\Omega \cap (\alpha((r, s)) \cup \beta((r, s))) = \emptyset$. By applying Lemma 7.11 to the paths $\alpha|_{[r, s]}$ and $\beta|_{[r, s]}$ and $\gamma = \alpha|_{[0, r]}$, we see that $[\alpha|_{[0, s]} \cdot \beta|_{[r, s]}^- \cdot \alpha|_{[0, r]}^-] \in N$.

A component (r, s) of $[0, 1] \setminus A$ is Type III if and only if $\Omega \cap (\alpha([r, s]) \cup \beta([r, s])) = \emptyset$. Hence, if (r, s) is Type III, the definition of A guarantees the existence of $q, t \in A$ such that (q, r) and (s, t) are Type I components and thus Ω intersects both $\alpha([q, r]) \cup \beta([q, r])$ and $\alpha([s, t]) \cup \beta([s, t])$. In particular r and s are isolated points of A . Let

$$A^* = A \setminus \{s \mid (r, s) \text{ is a Type III component of } [0, 1] \setminus A\}.$$

Since A^* is constructed by removing only isolated points of A , A^* is closed and nowhere dense. In particular, $[0, 1] \setminus A^*$ is formed by combining each type III component with a unique Type I component. It follows that for every component (r, s) of $[0, 1] \setminus A^*$, we have $\Omega \cap (\alpha([r, s]) \cup \beta([r, s])) \neq \emptyset$ and $[\alpha|_{[0, s]} \cdot \beta|_{[r, s]}^- \cdot \alpha|_{[0, r]}^-] \in N$.

Let \mathcal{C} denote the set of components of $[0, 1] \setminus A^*$ with the natural linear ordering inherited from $[0, 1]$. Recall that a subset $\mathcal{J} \subset \mathcal{C}$ is convex if whenever $I_1, I_2 \in \mathcal{J}$ and $I_1 < I < I_2$, then $I \in \mathcal{J}$. If $I = (r, s) \in \mathcal{C}$, define the *wild image* of I as the nonempty finite set $wim(I) = \Omega \cap (\alpha([r, s]) \cup \beta([r, s]))$. Since Ω is finite, the continuity of α and β guarantee that there can only be finitely many $I \in \mathcal{C}$ such that $|wim(I)| > 1$. Also note that any points from different sets $f^{-1}(x_0), f^{-1}(x_1), \dots, f^{-1}(x_n)$ are separated by a minimum distance. Therefore, we may write \mathcal{C} as a disjoint union of finitely many single-point sets $\{I_1\}, \dots, \{I_m\}$

with $|wim(I_j)| > 1$ and finitely many convex sets $\mathcal{J}_1, \dots, \mathcal{J}_{m'} \subseteq \mathcal{C}$ such that for each $i \in \{1, \dots, m'\}$, the interval \mathcal{J}_i has constant wild image of cardinality 1, i.e. there exists $x_k \in \Omega$ such that $I, I' \in \mathcal{J}_i \Rightarrow wim(I) = \{x_k\} = wim(I')$. Using this decomposition of \mathcal{C} , we may find points $0 = p_0 < p_1 < \dots < p_n = 1$ in A^* such that for each $j \in \{1, \dots, n\}$, either:

- (1) $(p_{j-1}, p_j) = I_j \in \mathcal{C}$ with $|wim(I_j)| > 1$ and thus $[\alpha|_{[0, p_j]} \cdot \beta|_{[p_{j-1}, p_j]} \cdot \alpha|_{[0, p_{j-1}]}] \in N$ by construction of A^*
- (2) or $\alpha|_{[p_{j-1}, p_j]}$ and $\beta|_{[p_{j-1}, p_j]}$ satisfy the conditions of Lemma 7.12 using the nowhere dense set $A^* \cap [p_{j-1}, p_j]$, path $\gamma = \alpha|_{[0, p_{j-1}]}$, and unique wild point $x_k \in \Omega$ such that $\Omega \cap (\alpha([p_{j-1}, p_j]) \cup \beta([p_{j-1}, p_j])) = \{x_k\}$. By applying this Lemma, we see that $[\alpha|_{[0, p_j]} \cdot \beta|_{[p_{j-1}, p_j]} \cdot \alpha|_{[0, p_{j-1}]}] \in N$.

Finally, since $[\alpha \cdot \beta]$ is a product of the elements $[\alpha|_{[0, p_j]} \cdot \beta|_{[p_{j-1}, p_j]} \cdot \alpha|_{[0, p_{j-1}]}]$, $1 \leq j \leq n$, we conclude that $[\alpha \cdot \beta^-] \in N$. □

Example 7.16. Since $\mathbf{w}(\mathbb{H}) = \{b_0\}$ is discrete, we may apply Theorem 7.13. Hence, if $N \trianglelefteq \pi_1(\mathbb{H}, b_0)$ is a normal subgroup, then $p_N : \tilde{\mathbb{H}}_N \rightarrow \mathbb{H}$ is a generalized regular covering if and only if \mathbb{H} has transfinite products relative to N .

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